Proof of Riemann Hypothesis for Dirichlet Beta using Hadamard Products

Abstract

(1) Completed Dirichlet Beta Function $\omega(z)$ and $\omega(1-z)$ are factored into the Hadamard products as follows.

$$\omega(z) = \Pi(1-z/\rho) \quad , \quad \omega(1-z) = \Pi(1-(1-z)/\rho)$$

(2) $\omega(z) = \omega(1-z)$ holds on the whole complex plane.

(3) $\Pi(1-z/\rho) = \Pi(1-(1-z)/\rho)$ holds on the whole complex plane if and only if all $Re(\rho)$ are $1/2$.

Because, assuming the existence of zeros off the critical line leads to a contradiction : the existence of two different Vieta's formulas with the same value. Thus, the Riemann Hypothesis holds as a theorem.

Introduction

Functions studied in this paper

In this paper, we study the Dirichlet Beta Function $\beta(z)$ and the Completed Dirichlet Beta Function $\omega(z)$, defined as follows.

(0.0) $$\beta(z) = \frac{1}{1^z} - \frac{1}{3^z} + \frac{1}{5^z} - \frac{1}{7^z} + \cdots = \frac{1}{4^z}\left\{\zeta\left(z, \frac{1}{4}\right) - \zeta\left(z, \frac{3}{4}\right)\right\}$$

(0.1) $$\omega(z) = \left(\frac{2}{\sqrt{\pi}}\right)^{1+z} \Gamma\left(\frac{1+z}{2}\right)\beta(z)$$

Hereafter, we will simply refer to these as the "beta function" and the "ω function" .

In addition, it is known that these zeros are equivalent in the critical strip ($0 < Re(z) < 1$) .

Notation for zeros of β(z) and ω(z)

The zeros ρ of the $\beta(z)$ and the $\omega(z)$ described in this paper as follows.

(1) Complex number notation

$$\sum_{k=1}^{\infty} \frac{1}{\rho_k} \quad , \quad \prod_{k=1}^{\infty}\left(1 - \frac{z}{\rho_k}\right)$$

(2) Real and imaginary parts notation

Putting $\rho_{2r-1} = x_r - iy_r$, $\rho_{2r} = x_r + iy_r$ $\quad r = 1, 2, 3, \cdots \quad (y_r > 0)$,

$$\sum_{r=1}^{\infty}\left(\frac{1}{x_r - iy_r} + \frac{1}{x_r + iy_r}\right) \quad , \quad \prod_{r=1}^{\infty}\left(1 - \frac{z}{x_r - iy_r}\right)\left(1 - \frac{z}{x_r + iy_r}\right)$$

1 ω Function and Hadamard Product

Theorem 1.1

Let the Completed Dirichlet Beta Function $\omega(z)$ be as follws.

(0.1) $$\omega(z) = \left(\frac{2}{\sqrt{\pi}}\right)^{1+z}\Gamma\left(\frac{1+z}{2}\right)\beta(z)$$

Then $\omega(z)$ and $\omega(1-z)$ are factorized by their zeros ρ_k $k=1,2,3,\cdots$ respectively as follows:

(1.1) $$\omega(z) = \prod_{k=1}^{\infty}\left(1-\frac{z}{\rho_k}\right)$$

(1.2) $$\omega(1-z) = \prod_{k=1}^{\infty}\left(1-\frac{1-z}{\rho_k}\right)$$

Proof

According to Formula 5.1.1 in my paper " **05 Factorization of Completed Dirichlet Beta** ", $\omega(z)$ is expressed as the Hadamard product as follows.

(9.1) $$\omega(z) = e^{\left(\frac{3\log\pi}{2}-\frac{\gamma}{2}-\log 2-4\log\Gamma\left(\frac{3}{4}\right)\right)z}\prod_{k=1}^{\infty}\left(1-\frac{z}{\rho_k}\right)\cdot e^{\sum_{k=1}^{\infty}\frac{z}{\rho_k}}$$

Here, let

$$\prod_{k=1}^{\infty}\left(1-\frac{z}{\rho_k}\right) = \prod_{r=1}^{\infty}\left(1-\frac{z}{x_r-iy_r}\right)\left(1-\frac{z}{x_r+iy_r}\right) = \prod_{r=1}^{\infty}\left(1-\frac{2x_r z}{x_r^2+y_r^2}+\frac{z^2}{x_r^2+y_r^2}\right)$$

$$\sum_{k=1}^{\infty}\frac{z}{\rho_k} = \sum_{r=1}^{\infty}\left(\frac{z}{x_r-iy_r}+\frac{z}{x_r+iy_r}\right) = \sum_{r=1}^{\infty}\frac{2x_r z}{x_r^2+y_r^2}$$

Then, (9.1) becoms

(9.1') $$\omega(z) = e^{\left(\frac{3\log\pi}{2}-\frac{\gamma}{2}-\log 2-4\log\Gamma\left(\frac{3}{4}\right)\right)z}\prod_{r=1}^{\infty}\left(1-\frac{2x_r z}{x_r^2+y_r^2}+\frac{z^2}{x_r^2+y_r^2}\right)\cdot e^{\sum_{r=1}^{\infty}\frac{2x_r z}{x_r^2+y_r^2}}$$

Furthermore, if we express x_n+iy_n $n=1,2,3,\cdots$ whose real parts are $1/2$ as $1/2\pm iy_r$ $r=1,2,3,\cdots$ and those whose real parts are not $1/2$ as $\alpha_s\pm i\beta_s$, $1-\alpha_s\pm i\beta_s$ $(0<\alpha_s<1/2\ ,\ \beta_s>0)$ $s=1,2,3,\cdots$ then (9.1') becomes

(9.1") $$\omega(z) = e^{\left(\frac{3\log\pi}{2}-\frac{\gamma}{2}-\log 2-4\log\Gamma\left(\frac{3}{4}\right)\right)z}\prod_{r=1}^{\infty}\left(1-\frac{z}{1/4+y_r^2}+\frac{z^2}{1/4+y_r^2}\right)\cdot e^{\sum_{r=1}^{\infty}\frac{z}{1/4+y_r^2}}$$

$$\times\prod_{s=1}\left(1-\frac{2\alpha_s z}{\alpha_s^2+\beta_s^2}+\frac{z^2}{\alpha_s^2+\beta_s^2}\right)\cdot e^{\sum_{s=1}\frac{2\alpha_s z}{\alpha_s^2+\beta_s^2}}$$

$$\times\prod_{s=1}\left\{1-\frac{2(1-\alpha_s)z}{(1-\alpha_s)^2+\beta_s^2}+\frac{z^2}{(1-\alpha_s)^2+\beta_s^2}\right\}\cdot e^{\sum_{s=1}\frac{2(1-\alpha_s)z}{(1-\alpha_s)^2+\beta_s^2}}$$

Substituting $z=1$ for both sides of (9.1') and (9.1") , (9.2') $$\omega(1) = 1 = e^{\frac{3\log\pi}{2} - \frac{\gamma}{2} - \log 2 - 4\log\Gamma\left(\frac{3}{4}\right)} \prod_{r=1}^{\infty}\left(1 - \frac{2x_r - 1}{x_r^2 + y_r^2}\right) \cdot e^{\sum_{r=1}^{\infty}\frac{2x_r}{x_r^2 + y_r^2}}$$

(9.2") $$\omega(1) = 1 = e^{\frac{3\log\pi}{2} - \frac{\gamma}{2} - \log 2 - 4\log\Gamma\left(\frac{3}{4}\right)} \times \prod_{s=1}\left(1 - \frac{2\alpha_s - 1}{\alpha_s^2 + \beta_s^2}\right)\left\{1 + \frac{2\alpha_s - 1}{(1-\alpha_s)^2 + \beta_s^2}\right\}$$
$$\times e^{\sum_{r=1}^{\infty}\frac{1}{1/4 + y_r^2} + \sum_{s=1}\left\{\frac{1-2\alpha_s}{(1/2-\alpha_s)^2+\beta_s^2} + \frac{1+2\alpha_s}{(1/2+\alpha_s)^2+\beta_s^2}\right\}}$$

From these,

(9.3) $$\prod_{r=1}^{\infty}\left(1 - \frac{2x_r - 1}{x_r^2 + y_r^2}\right) = \prod_{s=1}\left(1 - \frac{2\alpha_s - 1}{\alpha_s^2 + \beta_s^2}\right)\left\{1 + \frac{2\alpha_s - 1}{(1-\alpha_s)^2 + \beta_s^2}\right\}$$

(9.4) $$e^{\sum_{r=1}^{\infty}\frac{2x_r}{x_r^2 + y_r^2}} = e^{\sum_{r=1}^{\infty}\frac{1}{1/4 + y_r^2} + \sum_{s=1}\left\{\frac{2\alpha_s}{\alpha_s^2+\beta_s^2} + \frac{2(1-\alpha_s)}{(1-\alpha_s)^2+\beta_s^2}\right\}}$$

Here, conveniently,

$$\left(1 - \frac{2\alpha_s - 1}{\alpha_s^2 + \beta_s^2}\right)\left\{1 + \frac{2\alpha_s - 1}{(1-\alpha_s)^2 + \beta_s^2}\right\} = 1 - \frac{4\alpha_s^2 - 4\alpha_s + 1}{\left(\alpha_s^2+\beta_s^2\right)\left\{(1-\alpha_s)^2+\beta_s^2\right\}} - \frac{2\alpha_s - 1}{\alpha_s^2 + \beta_s^2} + \frac{2\alpha_s - 1}{(1-\alpha_s)^2 + \beta_s^2}$$

i.e.

$$\left(1 - \frac{2\alpha_s - 1}{\alpha_s^2 + \beta_s^2}\right)\left\{1 + \frac{2\alpha_s - 1}{(1-\alpha_s)^2 + \beta_s^2}\right\} = 1$$

Then, (9.3) , (9.4) becom

(9.3') $$\prod_{n=1}^{\infty}\left(1 - \frac{2x_n - 1}{x_n^2 + y_n^2}\right) = 1 \qquad \left\{\text{i.e. } \prod_{r=1}^{\infty}\left(1 - \frac{1}{x_r - iy_r}\right)\left(1 - \frac{1}{x_r + iy_r}\right) = \prod_{k=1}^{\infty}\left(1 - \frac{1}{\rho_k}\right) = 1\right\}$$

(9.4') $$\sum_{n=1}^{\infty}\frac{2x_n}{x_n^2 + y_n^2} = \sum_{r=1}^{\infty}\frac{1}{1/4 + y_r^2} + \sum_{s=1}\left\{\frac{2\alpha_s}{\alpha_s^2+\beta_s^2} + \frac{2(1-\alpha_s)}{(1-\alpha_s)^2+\beta_s^2}\right\}$$

Substituting (9.3') for (9.2') ,

$$\omega(1) = 1 = e^{\frac{3\log\pi}{2} - \frac{\gamma}{2} - \log 2 - 4\log\Gamma\left(\frac{3}{4}\right)} \cdot e^{\sum_{r=1}^{\infty}\frac{2x_r}{x_r^2 + y_r^2}} = e^{\frac{3\log\pi}{2} - \frac{\gamma}{2} - \log 2 - 4\log\Gamma\left(\frac{3}{4}\right)} e^{\sum_{k=1}^{\infty}\frac{1}{\rho_k}}$$

Since $1 = e^0$,

(9.5) $$\sum_{k=1}^{\infty}\frac{1}{\rho_k} = 4\log\Gamma\left(\frac{3}{4}\right) + \frac{\gamma}{2} + \log 2 - \frac{3\log\pi}{2} = 0.07778398\cdots$$

Therefore,

$$e^{\sum_{k=1}^{\infty}\frac{z}{\rho_k}} = e^{\left(4\log\Gamma\left(\frac{3}{4}\right) + \frac{\gamma}{2} + \log 2 - \frac{3\log\pi}{2}\right)z}$$

Substituting this for the right side of (9.1) ,

$$\omega(z) = e^{\left(\frac{3\log\pi}{2} - \frac{\gamma}{2} - \log 2 - 4\log\Gamma\left(\frac{3}{4}\right)\right)z} \prod_{k=1}^{\infty}\left(1 - \frac{z}{\rho_k}\right) \cdot e^{\left(4\log\Gamma\left(\frac{3}{4}\right) + \frac{\gamma}{2} + \log 2 - \frac{3\log\pi}{2}\right)z}$$

i.e.

(0.1) $$\omega(z) = \prod_{k=1}^{\infty}\left(1-\frac{z}{\rho_k}\right)$$

Since both sides are holomorphic on the whole complex plane, replacing z with $1-z$

(0.2) $$\omega(1-z) = \prod_{k=1}^{\infty}\left(1-\frac{1-z}{\rho_k}\right)$$

Q.E.D.

Note

When $\omega(z) = 1+A_1z^1+A_2z^2+A_3z^4+\cdots$, according to Vieta's formulas, the following holds.

(9.5) $$\sum_{k=1}^{\infty}\frac{1}{\rho_k} = -A_1 = -\left(\frac{3\log\pi}{2}-\frac{\gamma}{2}-\log 2-4\log\Gamma\left(\frac{3}{4}\right)\right) = -(-0.07778398\cdots)$$

2 Functional Equation

Theorem 2.1 (Functional Equation)

Let the Completed Dirichlet Beta Function $\omega(z)$ be as follws.

(0.1) $$\omega(z) = \left(\frac{2}{\sqrt{\pi}}\right)^{1+z} \Gamma\left(\frac{1+z}{2}\right)\beta(z)$$

Then, the following expression holds on the whole complex plane .

(2.1) $$\omega(z) = \omega(1-z)$$

Proof

The following functional equation is known for the Completed Dirichlet Beta Function $\omega(z)$.

$$\beta(z) = \left(\frac{2}{\pi}\right)^{1-z} \cos\frac{\pi z}{2}\,\Gamma(1-z)\beta(1-z) \qquad z \neq 1,2,3,\cdots$$

Here,

$$\cos\frac{\pi z}{2} = \frac{\pi}{\Gamma\left(\frac{1+z}{2}\right)\Gamma\left(\frac{1-z}{2}\right)}$$

Substituting this for the obove,

$$\Gamma\left(\frac{1+z}{2}\right)\beta(z) = \left(\frac{2}{\pi}\right)^{1-z} \frac{\pi\,\Gamma(1-z)}{\Gamma\left(\frac{1-z}{2}\right)}\beta(1-z)$$

Further,

$$\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right) \quad\Longrightarrow\quad \frac{\Gamma(1-z)}{\Gamma\left(\frac{1-z}{2}\right)} = \frac{2^{-z}}{\sqrt{\pi}}\Gamma\left(\frac{1}{2}+\frac{1-z}{2}\right)$$

Substituting this for the obove,

$$\begin{aligned}\Gamma\left(\frac{1+z}{2}\right)\beta(z) &= \left(\frac{2}{\pi}\right)^{1-z} \frac{2^{-z}\pi}{\sqrt{\pi}}\Gamma\left(\frac{1}{2}+\frac{1-z}{2}\right)\beta(1-z) \\ &= \frac{2^{1-z}}{(\sqrt{\pi})^{1-z}}\frac{2^{-z}\sqrt{\pi}}{(\sqrt{\pi})^{1-z}}\Gamma\left(\frac{1}{2}+\frac{1-z}{2}\right)\beta(1-z) \\ &= \left(\frac{2}{\sqrt{\pi}}\right)^{1-z}\left(\frac{2}{\sqrt{\pi}}\right)^{-z}\Gamma\left(\frac{1}{2}+\frac{1-z}{2}\right)\beta(1-z)\end{aligned}$$

Multiplying both sides by $(2/\sqrt{\pi})^{1+z}$,

$$\left(\frac{2}{\sqrt{\pi}}\right)^{1+z}\Gamma\left(\frac{1}{2}+\frac{z}{2}\right)\beta(z) = \left(\frac{2}{\sqrt{\pi}}\right)^{2-z}\Gamma\left(\frac{1}{2}+\frac{1-z}{2}\right)\beta(1-z)$$

i.e.

(2.1) $$\omega(z) = \omega(1-z)$$

Since $\omega(z)$ is holomorphic on the whole complex plane, this equation also holds on the whole complex plane.

Q.E.D.

3 Hadamard Product Equation

The equation $\Pi(1-z/\rho)=\Pi(1-(1-z)/\rho)$ for the Hadamard product does not hold generally. Because putting $\rho_{2r-1}=x_r-iy_r$, $\rho_{2r}=x_r+iy_r$ $r=1,2,3,\cdots$, this equation can be expressed as follows.

$$\prod_{r=1}^{\infty}\left(1-\frac{z}{x_r-iy_r}\right)\left(1-\frac{z}{x_r+iy_r}\right)=\prod_{r=1}^{\infty}\left(1-\frac{1-z}{x_r-iy_r}\right)\left(1-\frac{1-z}{x_r+iy_r}\right)$$

However, expanding the ( )( ) on both sides,

$$\prod_{r=1}^{\infty}\left(1-\frac{2x_r z}{x_r^2+y_r^2}+\frac{z^2}{x_r^2+y_r^2}\right)=\prod_{r=1}^{\infty}\left(1-\frac{2x_r-1}{x_r^2+y_r^2}-\frac{2(1-x_r)z}{x_r^2+y_r^2}+\frac{z^2}{x_r^2+y_r^2}\right)$$

Both sides are generally different power series. That is, this equality generally does not hold on the whole complex plane.

In this chapter, we will find the necessary and sufficient condition for $\Pi(1-z/\rho)=\Pi(1-(1-z)/\rho)$, thereby proving that the Riemann hypothesis for the Dirichlet Beta Function $\beta(z)$ holds as a theorem.

Theorem 3.1 (Functional Equation for Hadamard Products)

Let Completed Dirichlet Beta Function $\omega(z)$ and $\omega(1-z)$ are factorized by their zeros ρ_k $k=1,2,3,\cdots$ as follows.

(1.1) $$\omega(z)=\prod_{k=1}^{\infty}\left(1-\frac{z}{\rho_k}\right)$$

(1.2) $$\omega(1-z)=\prod_{k=1}^{\infty}\left(1-\frac{1-z}{\rho_k}\right)$$

Then, if and only if $Re(\rho_k)=1/2$ $k=1,2,3,\cdots$, The following equation holds on the whole complex plane.

(3.1) $$\prod_{k=1}^{\infty}\left(1-\frac{z}{\rho_k}\right)=\prod_{k=1}^{\infty}\left(1-\frac{1-z}{\rho_k}\right)$$

Proof

Express the zeros ρ_k as follows, with separate real and imaginary parts:

$$\rho_{2r-1}=x_r-iy_r\ ,\ \rho_{2r}=x_r+iy_r \quad r=1,2,3,\cdots \qquad (y_r>0)$$

Then (3.1) becomes

(3.1') $$\prod_{r=1}^{\infty}\left(1-\frac{z}{x_r-iy_r}\right)\left(1-\frac{z}{x_r+iy_r}\right)=\prod_{r=1}^{\infty}\left(1-\frac{1-z}{x_r-iy_r}\right)\left(1-\frac{1-z}{x_r+iy_r}\right)$$

I. Sufficiency

If $x_r=1/2$ $r=1,2,3,\cdots$, (3.1') becoms

(3.1") $$\prod_{r=1}^{\infty}\left(1-\frac{z}{1/2-iy_r}\right)\left(1-\frac{z}{1/2+iy_r}\right)=\prod_{r=1}^{\infty}\left(1-\frac{1-z}{1/2-iy_r}\right)\left(1-\frac{1-z}{1/2+iy_r}\right)$$

Expanding the ( )( ) on both sides,

(3.1") $$\prod_{r=1}^{\infty}\left(1-\frac{z}{1/4+y_r^2}+\frac{z^2}{1/4+y_r^2}\right)=\prod_{r=1}^{\infty}\left(1-\frac{z}{1/4+y_r^2}+\frac{z^2}{1/4+y_r^2}\right)$$

Since this is an identity, it holds on the whole complex plane. So, (3.1) holds on the whole complex plane.

The important thing here is that these zeros actually exist. (e.g. $1/2+i\,6.0209489\cdots$)

For these existing zerors, the following is completed from Theorem 1.1 and Theorem 2.1 .

$$\omega(z) = \prod_{k=1}^{\infty}\left(1-\frac{z}{\rho_k}\right) = \prod_{k=1}^{\infty}\left(1-\frac{1-z}{\rho_k}\right) = \omega(1-z)$$

As the result, the following equation holds from Vieta's formula (9.5) .

(9.5')
$$\sum_{r=1}^{\infty}\left(\frac{1}{1/2-iy_r}+\frac{1}{1/2+iy_r}\right) = 4\log\Gamma\left(\frac{3}{4}\right)+\frac{\gamma}{2}+\log 2-\frac{3\log\pi}{2} = 0.07778398\cdots$$

II. Necessity

Now, assume that in addition to the zeros on the critical line, there are also zeros off the critical line.

It is known that a set of such zeros should consist of the following four.

$$\alpha_s \pm i\beta_s \ ,\ 1-\alpha_s \pm i\beta_s \quad (0<\alpha_s<1/2\ ,\ \beta_s>0) \quad s=1,2,3,\cdots$$

And these also satisfy (3.1) on the whole complex plane. In fact,

$$\left(1-\frac{z}{\alpha_s-i\beta_s}\right)\left(1-\frac{z}{\alpha_s+i\beta_s}\right)\left(1-\frac{z}{1-\alpha_s-i\beta_s}\right)\left(1-\frac{z}{1-\alpha_s+i\beta_s}\right)$$

$$= 1 - \frac{\left(2\alpha_s-2\alpha_s^2+2\beta_s^2\right)z - \left(1+2\alpha_s-2\alpha_s^2+2\beta_s^2\right)z^2+2z^3-z^4}{\left(\alpha_s^2+\beta_s^2\right)\left\{(1-\alpha_s)^2+\beta_s^2\right\}}$$

$$= \left(1-\frac{1-z}{\alpha_s-i\beta_s}\right)\left(1-\frac{1-z}{\alpha_s+i\beta_s}\right)\left(1-\frac{1-z}{1-\alpha_s-i\beta_s}\right)\left(1-\frac{1-z}{1-\alpha_s+i\beta_s}\right)$$

Then, for existing zerors and assumed zeros, the following is completed from Theorem 1.1 and Theorem 2.1 .

$$\omega(z) = \prod_{k=1}^{\infty}\left(1-\frac{z}{\rho_k}\right) = \prod_{k=1}^{\infty}\left(1-\frac{1-z}{\rho_k}\right) = \omega(1-z)$$

As the result, the following equation holds from Vieta's formula (9.5) .

$$\sum_{r=1}^{\infty}\left(\frac{1}{1/2-iy_r}+\frac{1}{1/2+iy_r}\right)+\sum_{s=1}\left(\frac{1}{\alpha_s-i\beta_s}+\frac{1}{\alpha_s+i\beta_s}\right)$$
$$+\sum_{s=1}\left(\frac{1}{1-\alpha_s-i\beta_s}+\frac{1}{1-\alpha_s+i\beta_s}\right) = 4\log\Gamma\left(\frac{3}{4}\right)+\frac{\gamma}{2}+\log 2-\frac{3\log\pi}{2}$$

i.e.

(9.5")
$$\sum_{r=1}^{\infty}\left(\frac{1}{1/2-iy_r}+\frac{1}{1/2+iy_r}\right)+2\sum_{s=1}\left\{\frac{\alpha_s}{\alpha_s^2+\beta_s^2}+\frac{1-\alpha_s}{(1-\alpha_s)^2+\beta_s^2}\right\}$$
$$= 4\log\Gamma\left(\frac{3}{4}\right)+\frac{\gamma}{2}+\log 2-\frac{3\log\pi}{2} = 0.07778398\cdots$$

However, since $0<\alpha_s<1/2\ ,\ \beta_s>0$,

$$2\sum_{s=1}\left\{\frac{\alpha_s}{\alpha_s^2+\beta_s^2}+\frac{1-\alpha_s}{(1-\alpha_s)^2+\beta_s^2}\right\} > 0$$

So, (9.5") contradicts (9.5') . Therefore, the assumed zeros cannot exist.

Consequently, (3.1) holds only if $Re(\rho_k) = 1/2 \quad k=1,2,3,\cdots$

Q.E.D.

Conclusion

According to the 3 Theorems above, only zeros on the critical line exist in the critical strip $(0 < Re(z) < 1)$.
Thus, the Riemann Hypothesis for the Dirichlet Beta Function $\beta(z)$ holds as a theorem.

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