Proof of the Riemann Hypothesis using Hadamard Products

Abstract

(1) Riemann xi functkion $\xi(z)$ and $\xi(1-z)$ are factored into the Hadamard products as follows.

$$\xi(z) = \Pi\,(1-z/\rho) \quad , \quad \xi(1-z) = \Pi\,(1-(1-z)/\rho)$$

(2) $\xi(z) = \xi(1-z)$ holds on the whole complex plane..

(3) $\Pi\,(1-z/\rho) = \Pi(1-(1-z)/\rho)$ holds on the whole complex plane if and only if all $Re(\rho)$ are $1/2$.

Because, assuming the existence of zeros off the critical line leads to a contradiction : the existence of two different Vieta's formulas with the same value. Thus, the Riemann Hypothesis holds as a theorem.

Introduction

Functions studied in this paper

In this paper, we study the Riemann zeta function and the Riemann xi function, which are defined as follows.

(0.0) $$\zeta(z) = \frac{1}{1^z} + \frac{1}{2^z} + \frac{1}{3^z} + \frac{1}{4^z} + \cdots$$

(0.1) $$\xi(z) = -z(1-z)\pi^{-\frac{z}{2}}\Gamma\left(\frac{z}{2}\right)\zeta(z)$$

Hereafter, we will simply refer to these as the "zeta function" and the "xi function" .

In addition, it is known that these zeros are equivalent in the critical strip ($0 < Re(z) < 1$) .

Notation for zeros of ζ(z) and ξ(z)

The zeros ρ of the zeta function $\zeta(z)$ and the xi function $\xi(z)$ are usually written as follows.

$$\sum_{\rho} \frac{1}{\rho} \quad , \quad \prod_{\rho}\left(1-\frac{z}{\rho}\right) \qquad \text{where } \rho \text{ runs over all the zeros.}$$

However, this notation is conceptual and vague, and cannot be used for practical calculations.

(1) Complex number notation

In 1914, Hardy and Littlewood proved that there are an infinite number of zeros on the critical line. This means that there are an infinite number of zeros in the critical strip. So, in this paper, we use the following notation.

$$\sum_{k=1}^{\infty} \frac{1}{\rho_k} \quad , \quad \prod_{k=1}^{\infty}\left(1-\frac{z}{\rho_k}\right)$$

(2) Real and imaginary parts notation

However, even with the notation (1), it is difficult to examine the real and imaginary parts of the zeros ρ_k in detail. So, considering that the xi function has conjugate zeros, we replace ρ_k $k=1,2,3,\cdots$ as follows.

$$\rho_{2r-1} = x_r - iy_r \; , \; \rho_{2r} = x_r + iy_r \qquad r=1,2,3,\cdots \qquad (y_r > 0)$$

Using this, the example (1) can be rewritten as follows.

$$\sum_{r=1}^{\infty}\left(\frac{1}{x_r - iy_r} + \frac{1}{x_r + iy_r}\right) \quad , \quad \prod_{r=1}^{\infty}\left(1-\frac{z}{x_r - iy_r}\right)\left(1-\frac{z}{x_r + iy_r}\right)$$

1 xi Function and Hadamard Product

Theorem 1.1

Let the Completed Riemann Zeta Function $\xi(z)$ be as follws.

(0.1) $$\xi(z) = -z(1-z)\pi^{-\frac{z}{2}}\Gamma\left(\frac{z}{2}\right)\zeta(z)$$

Then $\xi(z)$ and $\xi(1-z)$ are factorized by their zeros ρ_k $k=1,2,3,\cdots$ respectively as follows:

(1.1) $$\xi(z) = \prod_{k=1}^{\infty}\left(1-\frac{z}{\rho_k}\right)$$

(1.2) $$\xi(1-z) = \prod_{k=1}^{\infty}\left(1-\frac{1-z}{\rho_k}\right)$$

Proof

In 1893, ***Hadamard*** proved the following theorem.

$$\zeta(z) = \frac{(2\pi/e)^z}{2(z-1)}\prod_{k=1}^{\infty}\left(1-\frac{z}{\rho_k}\right)e^{\frac{z}{\rho_k}}\prod_{n=1}^{\infty}\left(1+\frac{z}{2n}\right)e^{-\frac{z}{2n}}$$

The ***Weierstrass*** expression for the gamma function is

$$\prod_{n=1}^{\infty}\left(1+\frac{z}{2n}\right)e^{-\frac{z}{2n}} = \frac{e^{-\gamma z/2}}{\Gamma(1+z/2)}$$

Substituting this for the right hand side of the above equation,

$$\zeta(z) = \frac{(2\pi/e)^z}{2(z-1)}\frac{e^{-\gamma z/2}}{\Gamma(1+z/2)}\prod_{k=1}^{\infty}\left(1-\frac{z}{\rho_k}\right)e^{\frac{z}{\rho_k}}$$

$$= \frac{e^{z\log 2\pi}e^{-z}}{z-1}\frac{e^{-\gamma z/2}}{2\Gamma(1+z/2)}\prod_{k=1}^{\infty}\left(1-\frac{z}{\rho_k}\right)e^{\frac{z}{\rho_k}}$$

$$= \frac{1}{(z-1)z\Gamma(z/2)}e^{\left(\log 2\pi-1-\frac{\gamma}{2}\right)z}\prod_{k=1}^{\infty}\left(1-\frac{z}{\rho_k}\right)e^{\frac{z}{\rho_k}}$$

From this,

$$-z(1-z)\Gamma\left(\frac{z}{2}\right)\zeta(z) = e^{\left(\log 2\pi-1-\frac{\gamma}{2}\right)z}\prod_{k=1}^{\infty}\left(1-\frac{z}{\rho_k}\right)e^{\frac{z}{\rho_k}}$$

Multiplying both sides by $\pi^{-z/2} = e^{-(z\log\pi)/2}$,

$$-z(1-z)\pi^{-\frac{z}{2}}\Gamma\left(\frac{z}{2}\right)\zeta(z) = e^{\left(\log 2\pi-\frac{\log\pi}{2}-1-\frac{\gamma}{2}\right)z}\prod_{k=1}^{\infty}\left(1-\frac{z}{\rho_k}\right)e^{\frac{z}{\rho_k}}$$

i.e.

(9.1) $$\xi(z) = e^{\left(\log 2+\frac{\log\pi}{2}-1-\frac{\gamma}{2}\right)z}\prod_{k=1}^{\infty}\left(1-\frac{z}{\rho_k}\right)e^{\sum_{k=1}^{\infty}\frac{z}{\rho_k}}$$

Here, let

$$\prod_{k=1}^{\infty}\left(1-\frac{z}{\rho_k}\right)=\prod_{r=1}^{\infty}\left(1-\frac{z}{x_r-iy_r}\right)\left(1-\frac{z}{x_r+iy_r}\right)=\prod_{r=1}^{\infty}\left(1-\frac{2x_r z}{x_r^2+y_r^2}+\frac{z^2}{x_r^2+y_r^2}\right)$$

$$\sum_{k=1}^{\infty}\frac{z}{\rho_k}=\sum_{r=1}^{\infty}\left(\frac{z}{x_r-iy_r}+\frac{z}{x_r+iy_r}\right)=\sum_{r=1}^{\infty}\frac{2x_r z}{x_r^2+y_r^2}$$

Then, (9.1) becomes

(9.1') $$\xi(z)=e^{\left(\log 2+\frac{\log\pi}{2}-1-\frac{\gamma}{2}\right)z}\prod_{r=1}^{\infty}\left(1-\frac{2x_r z}{x_r^2+y_r^2}+\frac{z^2}{x_r^2+y_r^2}\right)\cdot e^{\sum_{r=1}^{\infty}\frac{2x_r z}{x_r^2+y_r^2}}$$

Furthermore, if we express $x_n+iy_n \quad n=1,2,3,\cdots$ whose real parts are $1/2$ as $1/2\pm iy_r \quad r=1,2,3,\cdots$ and those whose real parts are not $1/2$ as $\alpha_s\pm i\beta_s$, $1-\alpha_s\pm i\beta_s \quad (0<\alpha_s<1/2 \ , \ \beta_s>0) \quad s=1,2,3,\cdots$ then (9.1') becomes

(9.1") $$\xi(z)=e^{\left(\log 2+\frac{\log\pi}{2}-1-\frac{\gamma}{2}\right)z}\prod_{r=1}^{\infty}\left(1-\frac{z}{1/4+y_r^2}+\frac{z^2}{1/4+y_r^2}\right)\cdot e^{\sum_{r=1}^{\infty}\frac{z}{1/4+y_r^2}}$$

$$\times\prod_{s=1}\left(1-\frac{2\alpha_s z}{\alpha_s^2+\beta_s^2}+\frac{z^2}{\alpha_s^2+\beta_s^2}\right)\cdot e^{\sum_{s=1}\frac{2\alpha_s z}{\alpha_s^2+\beta_s^2}}$$

$$\times\prod_{s=1}\left\{1-\frac{2(1-\alpha_s)z}{(1-\alpha_s)^2+\beta_s^2}+\frac{z^2}{(1-\alpha_s)^2+\beta_s^2}\right\}\cdot e^{\sum_{s=1}\frac{2(1-\alpha_s)z}{(1-\alpha_s)^2+\beta_s^2}}$$

Substituting $z=1$ for both sides of (9.1') and (9.1") ,

(9.2') $$\xi(1)=1=e^{\log 2+\frac{\log\pi}{2}-1-\frac{\gamma}{2}}\prod_{r=1}^{\infty}\left(1-\frac{2x_r-1}{x_r^2+y_r^2}\right)\cdot e^{\sum_{r=1}^{\infty}\frac{2x}{x_r^2+y_r^2}}$$

(9.2") $$\xi(1)=1=e^{\log 2+\frac{\log\pi}{2}-1-\frac{\gamma}{2}}\times\prod_{s=1}\left(1-\frac{2\alpha_s-1}{\alpha_s^2+\beta_s^2}\right)\left\{1+\frac{2\alpha_s-1}{(1-\alpha_s)^2+\beta_s^2}\right\}$$

$$\times e^{\sum_{r=1}^{\infty}\frac{1}{1/4+y_r^2}+\sum_{s=1}\left\{\frac{2\alpha_s}{\alpha_s^2+\beta_s^2}+\frac{2(1-\alpha_s)}{(1-\alpha_s)^2+\beta_s^2}\right\}}$$

From these,

(9.3) $$\prod_{r=1}^{\infty}\left(1-\frac{2x_r-1}{x_r^2+y_r^2}\right)=\prod_{s=1}\left(1-\frac{2\alpha_s-1}{\alpha_s^2+\beta_s^2}\right)\left\{1+\frac{2\alpha_s-1}{(1-\alpha_s)^2+\beta_s^2}\right\}$$

(9.4) $$e^{\sum_{r=1}^{\infty}\frac{2x_r}{x_r^2+y_r^2}}=e^{\sum_{r=1}^{\infty}\frac{1}{1/4+y_r^2}+\sum_{s=1}\left\{\frac{2\alpha_s}{\alpha_s^2+\beta_s^2}+\frac{2(1-\alpha_s)}{(1-\alpha_s)^2+\beta_s^2}\right\}}$$

Here, conveniently,

$$\left(1-\frac{2\alpha_s-1}{\alpha_s^2+\beta_s^2}\right)\left\{1+\frac{2\alpha_s-1}{(1-\alpha_s)^2+\beta_s^2}\right\}=1-\frac{4\alpha_s^2-4\alpha_s+1}{\left(\alpha_s^2+\beta_s^2\right)\left\{(1-\alpha_s)^2+\beta_s^2\right\}}$$

$$-\frac{2\alpha_s-1}{\alpha_s^2+\beta_s^2}+\frac{2\alpha_s-1}{(1-\alpha_s)^2+\beta_s^2}$$

i.e.

$$\left(1-\frac{2\alpha_s-1}{\alpha_s^2+\beta_s^2}\right)\left\{1+\frac{2\alpha_s-1}{(1-\alpha_s)^2+\beta_s^2}\right\} = 1$$

Then, (9.3) , (9.4) becom

(9.3') $$\prod_{n=1}^{\infty}\left(1-\frac{2x_n-1}{x_n^2+y_n^2}\right) = 1 \qquad \left\{\text{i.e. } \prod_{r=1}^{\infty}\left(1-\frac{1}{x_r-iy_r}\right)\left(1-\frac{1}{x_r+iy_r}\right) = \prod_{k=1}^{\infty}\left(1-\frac{1}{\rho_k}\right) = 1\right\}$$

(9.4') $$\sum_{n=1}^{\infty}\frac{2x_n}{x_n^2+y_n^2} = \sum_{r=1}^{\infty}\frac{1}{1/4+y_r^2}+\sum_{s=1}\left\{\frac{2\alpha_s}{\alpha_s^2+\beta_s^2}+\frac{2(1-\alpha_s)}{(1-\alpha_s)^2+\beta_s^2}\right\}$$

Substituting (9.3') for (9.2') ,

$$\xi(1) = 1 = e^{\log 2+\frac{\log\pi}{2}-1-\frac{\gamma}{2}}\cdot e^{\sum_{r=1}^{\infty}\frac{2x}{x_r^2+y_r^2}} = e^{\log 2+\frac{\log\pi}{2}-1-\frac{\gamma}{2}}e^{\sum_{k=1}^{\infty}\frac{1}{\rho_k}}$$

Since $1 = e^0$,

(9.5) $$\sum_{k=1}^{\infty}\frac{1}{\rho_k} = 1+\frac{\gamma}{2}-\log 2-\frac{\log\pi}{2} = 0.0230957\cdots$$

Therefore,

$$e^{\sum_{k=1}^{\infty}\frac{z}{\rho_k}} = e^{\left(1+\frac{\gamma}{2}-\log 2-\frac{\log\pi}{2}\right)z}$$

Substituting this for the right side of (9.1) ,

$$\xi(z) = e^{\left(\log 2+\frac{\log\pi}{2}-1-\frac{\gamma}{2}\right)z}\prod_{k=1}^{\infty}\left(1-\frac{z}{\rho_k}\right)\cdot e^{\left(1+\frac{\gamma}{2}-\log 2-\frac{\log\pi}{2}\right)z}$$

i.e.

(0.1) $$\xi(z) = \prod_{k=1}^{\infty}\left(1-\frac{z}{\rho_k}\right)$$

Since both sides are holomorphic on the whole complex plane, replacing z with $1-z$

(0.2) $$\xi(1-z) = \prod_{k=1}^{\infty}\left(1-\frac{1-z}{\rho_k}\right)$$

Q.E.D.

Note

When $\xi(z) = 1+A_1z^1+A_2z^2+A_3z^4+\cdots$, according to Vieta's formulas, the following holds.

(9.5) $$\sum_{k=1}^{\infty}\frac{1}{\rho_k} = -A_1 = -\left(\log 2+\frac{\log\pi}{2}-1-\frac{\gamma}{2}\right) = -(-0.0230957\cdots)$$

2 Functional Equation

Theorem 2.1 (Functional Equation)

Let xi function be

(0.1) $$\xi(z) = -z(1-z)\pi^{-\frac{z}{2}}\Gamma\left(\frac{z}{2}\right)\zeta(z)$$

Then, the following expression holds on the whole complex plane .

(2.1) $$\xi(z) = \xi(1-z)$$

Proof

According to ***Riemann***, the following holds except for two points on the complex plane .

$$\pi^{-\frac{z}{2}}\Gamma\left(\frac{z}{2}\right)\zeta(z) = \pi^{-\frac{1-z}{2}}\Gamma\left(\frac{1-z}{2}\right)\zeta(1-z) \qquad Re(z) \ne 0,1$$

$$z(1-z) = (1-z)\{1-(1-z)\}$$

Substituting these for the right side of (0.1) ,

$$\xi(z) = -(1-z)\{1-(1-z)\}\pi^{-\frac{1-z}{2}}\Gamma\left(\frac{1-z}{2}\right)\zeta(1-z) = \xi(1-z)$$

Since $\xi(z)$ is holomorphic on the whole complex plane, the fnctional equation holds on the whole complex plane.

Q.E.D.

3 Hadamard Product Equation

The equation $\prod(1-z/\rho)=\prod(1-(1-z)/\rho)$ for the Hadamard product does not hold generally. Because putting $\rho_{2r-1}=x_r-iy_r$, $\rho_{2r}=x_r+iy_r$ $r=1,2,3,\cdots$, this equation can be expressed as follows.

$$\prod_{r=1}^{\infty}\left(1-\frac{z}{x_r-iy_r}\right)\left(1-\frac{z}{x_r+iy_r}\right)=\prod_{r=1}^{\infty}\left(1-\frac{1-z}{x_r-iy_r}\right)\left(1-\frac{1-z}{x_r+iy_r}\right)$$

However, expanding the ( )( ) on both sides,

$$\prod_{r=1}^{\infty}\left(1-\frac{2x_r z}{x_r^2+y_r^2}+\frac{z^2}{x_r^2+y_r^2}\right)=\prod_{r=1}^{\infty}\left(1-\frac{2x_r-1}{x_r^2+y_r^2}-\frac{2(1-x_r)z}{x_r^2+y_r^2}+\frac{z^2}{x_r^2+y_r^2}\right)$$

Both sides are generally different power series. That is, this equality generally does not hold on the whole complex plane.

In this chapter, we will find the necessary and sufficient condition for $\prod(1-z/\rho)=\prod(1-(1-z)/\rho)$, thereby proving that the Riemann hypothesis holds as a theorem.

Theorem 3.1 (Functional Equation for Hadamard Products)

Let the xi Function $\xi(z)$ and $\xi(1-z)$ are factorized by their zeros ρ_k $k=1,2,3,\cdots$ as follows.

(1.1) $$\xi(z)=\prod_{k=1}^{\infty}\left(1-\frac{z}{\rho_k}\right)$$

(1.2) $$\xi(1-z)=\prod_{k=1}^{\infty}\left(1-\frac{1-z}{\rho_k}\right)$$

Then, if and only if $Re(\rho_k)=1/2$ $k=1,2,3,\cdots$, The following equation holds on the whole complex plane.

(3.1) $$\prod_{k=1}^{\infty}\left(1-\frac{z}{\rho_k}\right)=\prod_{k=1}^{\infty}\left(1-\frac{1-z}{\rho_k}\right)$$

Proof

Express the zeros ρ_k as follows, with separate real and imaginary parts:

$$\rho_{2r-1}=x_r-iy_r\ ,\ \rho_{2r}=x_r+iy_r\quad (y_r>0)\qquad r=1,2,3,\cdots$$

Then (3.1) becomes

(3.1') $$\prod_{r=1}^{\infty}\left(1-\frac{z}{x_r-iy_r}\right)\left(1-\frac{z}{x_r+iy_r}\right)=\prod_{r=1}^{\infty}\left(1-\frac{1-z}{x_r-iy_r}\right)\left(1-\frac{1-z}{x_r+iy_r}\right)$$

I. Sufficiency

If $x_r=1/2$ $r=1,2,3,\cdots$, (3.1') becoms

(3.1") $$\prod_{r=1}^{\infty}\left(1-\frac{z}{1/2-iy_r}\right)\left(1-\frac{z}{1/2+iy_r}\right)=\prod_{r=1}^{\infty}\left(1-\frac{1-z}{1/2-iy_r}\right)\left(1-\frac{1-z}{1/2+iy_r}\right)$$

Expanding the ( )( ) on both sides,

(3.1") $$\prod_{r=1}^{\infty}\left(1-\frac{z}{1/4+y_r^2}+\frac{z^2}{1/4+y_r^2}\right)=\prod_{r=1}^{\infty}\left(1-\frac{z}{1/4+y_r^2}+\frac{z^2}{1/4+y_r^2}\right)$$

Since this is an identity, it holds on the whole complex plane. So, (3.1) holds on the whole complex plane.

The important thing here is that these zeros actually exist. (e.g. $1/2+i\,14.13472514\cdots$)

For these existing zerors, the following is completed from Theorem 1.1 and Theorem 2.1 .

$$\xi(z) = \prod_{k=1}^{\infty}\left(1-\frac{z}{\rho_k}\right) = \prod_{k=1}^{\infty}\left(1-\frac{1-z}{\rho_k}\right) = \xi(1-z)$$

As the result, the following equation holds from Vieta's formula (9.5) .

(9.5')
$$\sum_{r=1}^{\infty}\left(\frac{1}{1/2-iy_r}+\frac{1}{1/2+iy_r}\right) = 1+\frac{\gamma}{2} - \log 2 - \frac{\log\pi}{2} = 0.0230957\cdots$$

II. Necessity

Now, assume that in addition to the zeros on the critical line, there are also zeros off the critical line.

It is known that a set of such zeros should consist of the following four.

$$\alpha_s \pm i\beta_s \ ,\ 1-\alpha_s \pm i\beta_s \quad \left(0<\alpha_s<1/2\ ,\ \beta_s>0\right) \quad s=1,2,3,\cdots$$

And these also satisfy (3.1) on the whole complex plane. In fact,

$$\left(1-\frac{z}{\alpha_s-i\beta_s}\right)\left(1-\frac{z}{\alpha_s+i\beta_s}\right)\left(1-\frac{z}{1-\alpha_s-i\beta_s}\right)\left(1-\frac{z}{1-\alpha_s+i\beta_s}\right)$$

$$= 1 - \frac{\left(2\alpha_s-2\alpha_s^2+2\beta_s^2\right)z - \left(1+2\alpha_s-2\alpha_s^2+2\beta_s^2\right)z^2 + 2z^3 - z^4}{\left(\alpha_s^2+\beta_s^2\right)\left\{\left(1-\alpha_s\right)^2+\beta_s^2\right\}}$$

$$= \left(1-\frac{1-z}{\alpha_s-i\beta_s}\right)\left(1-\frac{1-z}{\alpha_s+i\beta_s}\right)\left(1-\frac{1-z}{1-\alpha_s-i\beta_s}\right)\left(1-\frac{1-z}{1-\alpha_s+i\beta_s}\right)$$

Then, for existing zerors and assumed zeros, the following is completed from Theorem 1.1 and Theorem 2.1 .

$$\xi(z) = \prod_{k=1}^{\infty}\left(1-\frac{z}{\rho_k}\right) = \prod_{k=1}^{\infty}\left(1-\frac{1-z}{\rho_k}\right) = \xi(1-z)$$

As the result, the following equation holds from Vieta's formula (9.5) .

$$\sum_{r=1}^{\infty}\left(\frac{1}{1/2-iy_r}+\frac{1}{1/2+iy_r}\right) + \sum_{s=1}\left(\frac{1}{\alpha_s-i\beta_s}+\frac{1}{\alpha_s+i\beta_s}\right)$$

$$+\sum_{s=1}\left(\frac{1}{1-\alpha_s-i\beta_s}+\frac{1}{1-\alpha_s+i\beta_s}\right) = 1+\frac{\gamma}{2} - \log 2 - \frac{\log\pi}{2}$$

i.e.

(9.5")
$$\sum_{r=1}^{\infty}\left(\frac{1}{1/2-iy_r}+\frac{1}{1/2+iy_r}\right) + 2\sum_{s=1}\left\{\frac{\alpha_s}{\alpha_s^2+\beta_s^2}+\frac{1-\alpha_s}{\left(1-\alpha_s\right)^2+\beta_s^2}\right\}$$

$$= 1+\frac{\gamma}{2} - \log 2 - \frac{\log\pi}{2} = 0.0230957\cdots$$

However, since $0<\alpha_s<1/2\ ,\ \beta_s>0$,

$$2\sum_{s=1}\left\{\frac{\alpha_s}{\alpha_s^2+\beta_s^2}+\frac{1-\alpha_s}{\left(1-\alpha_s\right)^2+\beta_s^2}\right\} > 0$$

So, (9.5") contradicts (9.5') . Therefore, the assumed zeros cannot exist.

Consequently, (3.1) holds only if $Re\left(\rho_k\right) = 1/2 \quad k=1,2,3,\cdots$

Q.E.D.

Conclusion

According to the 3 Theorems above, only zeros on the critical line exist in the critical strip $(0 < Re(z) < 1)$.
.Thus, the Riemann Hypothesis holds as a theorem.

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