**Proof of the Riemann Hypothesis using Li Coefficients**


## Abstract

**(1)** The Li coefficients can be defined at the left edge of the critical strip. And this can be derived from both the xi function and the Hadamard product. If these are written as oλn and oμn respectively, then oλn = oμn must be true.

**(2)** The Li coefficients can also be defined at the right edge of the critical strip. And this can also be derived from both the xi function and the Hadamard product. If these are written as 1λn and 1μn respectively, then 1λn = 1μn must be true.

**(3)** The Li coefficients from the xi function becomes oλn = 1λn by functional equation and the definition. On the other hand, the Li coefficients from the Hadamard product is generally oμn ≠ 1μn. However, if the zeros of the xi function are on the critical line, then identically oμn=1μn. In this case, they are expressed as sums of squares of real numbers. That is, Li's criterion is satisfied. Thus, if the zeros of the xi function lie on the critical line, the system of equations is completed.

**(4)** If we assume that there are zeros outside the critical line, a contradiction occurs with the result of **(3)** . Therefore, there are no zeros outside the critical line, and the Riemann hypothesis is established as a theorem.


## 1 Li Coefficients at the Left Edge of the Critical Strip

The Li coefficients in this chapter are defined at the left edge of the critical strip $z=0$ .

### 1.1 Li Coefficients from the xi function oλn

The Li coefficients in this section are derived from the Riemann xi function.

**Theorem 1.1**

Let us define the Li coefficient ${}_0\lambda_n$ using the following two formulas.

$$ {}_0\lambda_n = \frac{(-1)^n}{(n-1)!}\left[\frac{d^n}{dz^n}(1-z)^{n-1}\log\xi(z)\right]_{z=0} \tag{1.1d}$$

$$ \xi(z) = -z(1-z)\pi^{-\frac{z}{2}}\Gamma\left(\frac{z}{2}\right)\zeta(z) \tag{0.ξ}$$

Then, ${}_0\lambda_n$ is expressed as follows:

$$ {}_0\lambda_n = \frac{1}{(n-1)!}\sum_{s=1}^{n}\binom{n}{s}(s)_{n-s}(-1)^s\left[\sum_{t=1}^{s}(-1)^{t-1}(t-1)!\frac{B_{s,t}\left(\xi^{(1)},\xi^{(2)},\cdots,\xi^{(s)}\right)}{\xi(z)^t}\right]_{z=0} \tag{1.λ}$$

Where, $(a)_k$ is the Pochhammer symbol and $B_{n,k}(f_1, f_2, \ldots)$ is the Bell polynomial.

**Proof**

**(1) Higher-order Derivatives of $(1-z)^{n-1}$**

$$\frac{d^1}{dz^1}(1-z)^{n-1} = -(n-1)(1-z)^{n-2}$$

$$\frac{d^2}{dz^2}(1-z)^{n-1} = (n-2)(n-1)(1-z)^{n-3}$$

$$\frac{d^3}{dz^3}(1-z)^{n-1} = -(n-3)(n-2)(n-1)(1-z)^{n-4}$$

$$\vdots$$

$$\frac{d^s}{dz^s}(1-z)^{n-1} = (-1)^s\{(n-s)\cdots(n-2)(n-1)\}(1-z)^{n-1-s}$$

Pochhammer symbol is

$$(a)_k = a(a+1)\cdots(a+k-1)$$

Using this,

$$(n-s)_k = (n-s)(n-s+1)\cdots(n-s+k-1)$$

From this,

$$(n-s)_s = (n-s)(n-s+1)\cdots(n-1) = (n-s)\ \cdots\ (n-2)(n-1)$$

Therefore,

$$\frac{d^s}{dz^s}(1-z)^{n-1} = (-1)^s (n-s)_s (1-z)^{n-1-s}$$

Replacing $s$ with $n-s$ ,

$$\frac{d^{n-s}}{dz^{n-s}}(1-z)^{n-1} = (-1)^{n-s}(s)_{n-s}(1-z)^{s-1} \tag{1.1}$$

**(2) Higher-order Derivatives of $log\ \xi(z)$**

According to **22.2.3** in my paper "**22 Higher Derivative of Composition**" , when $B_{s,t}(f_1, f_2, \ldots)$ is Bell polynomial

$$\{log\ f(x)\}^{(n)} = \sum_{r=1}^{n}(-1)^{r-1}(r-1)!\,B_{n,r}(f_1, f_2, \ldots, f_n)f^{-r} \qquad n \ge 1$$

So,

$$\frac{d^s}{dz^s}log\ \xi(z) = \sum_{t=1}^{s}(-1)^{t-1}(t-1)!\,\frac{B_{s,t}\left(\xi^{(1)}, \xi^{(2)}, \cdots, \xi^{(s)}\right)}{\xi(z)^t} \qquad s \ge 1 \tag{1.2$\lambda$}$$

**(3) Higher-order Derivatives of $(1-z)^{n-1}log\ \xi(z)$**

Leibniz's law is

$$\{f(z)g(z)\}^{(n)} = \sum_{s=0}^{n}\binom{n}{s}f^{(n-s)}(z)g^{(s)}(z)$$

Substitute (1.1) and (1.2$\lambda$) for this. Then since $n,s \ge 1$ ,

$$\frac{d^n}{dz^n}(1-z)^{n-1}log\ \xi(z) = \sum_{s=1}^{n}\binom{n}{s}\left\{(-1)^{n-s}(s)_{n-s}(1-z)^{s-1}\right\}$$
$$\times\sum_{t=1}^{s}(-1)^{t-1}(t-1)!\,\frac{B_{s,t}\left(\xi^{(1)}, \xi^{(2)}, \cdots, \xi^{(s)}\right)}{\xi(z)^t} \tag{1.3$\lambda$}$$

**(4) Li Coefficients ${}_0\lambda_n$**

Substituting (1.3$\lambda$) for (1.1d) ,

$${}_0\lambda_n = \frac{(-1)^n}{(n-1)!}\left[\sum_{s=1}^{n}\binom{n}{s}\left\{(-1)^{n-s}(s)_{n-s}(1-z)^{s-1}\right\}\right.$$
$$\left.\times\sum_{t=1}^{s}(-1)^{t-1}(t-1)!\,\frac{B_{s,t}\left(\xi^{(1)}, \xi^{(2)}, \cdots, \xi^{(s)}\right)}{\xi(z)^t}\right]_{z=0}$$

i.e.

$${}_0\lambda_n = \frac{1}{(n-1)!}\sum_{s=1}^{n}\binom{n}{s}(s)_{n-s}(-1)^s\left[\sum_{t=1}^{s}(-1)^{t-1}(t-1)!\,\frac{B_{s,t}\left(\xi^{(1)}, \xi^{(2)}, \cdots, \xi^{(s)}\right)}{\xi(z)^t}\right]_{z=0} \tag{1.$\lambda$}$$

Q.E.D.

The first few are

$$_0\lambda_1 = \frac{(-1)^1}{0!}\lim_{z\to 0}\frac{\xi'(z)}{\xi(z)}$$

$$_0\lambda_2 = \frac{(-1)^2}{1!}\lim_{z\to 0}\left\{-\frac{2\xi'(z)}{\xi(z)} - \frac{(1-z)\xi'(z)^2}{\xi(z)^2} + \frac{(1-z)\xi''(z)}{\xi(z)}\right\}$$

$$_0\lambda_3 = \frac{(-1)^3}{2!}\lim_{z\to 0}\left\{\frac{6\xi'(z)}{\xi(z)} + \frac{6(1-z)\xi'(z)^2}{\xi(z)^2} + \frac{2(1-z)^2\xi'(z)^3}{\xi(z)^3} - \frac{6(1-z)\xi''(z)}{\xi(z)} - \frac{3(1-z)^2\xi'(z)\xi''(z)}{\xi(z)^2} + \frac{(1-z)^2\xi^{(3)}(z)}{\xi(z)}\right\}$$

$$\vdots$$

Further, when we calculate these using the mathematical processing software *Mathematica*, it is as follows.

```
f[n_, z_] := (1 - z)^(n-1) Log[ξ[z]]
ξ[z_] := -z (1 - z) π^(-z/2) Gamma[z/2] Zeta[z]
oλ1 := (-1)^1/0! Limit[FullSimplify[∂z f[1, z]], z → 0]
N[oλ1]    0.02309570896612101`
oλ2 := (-1)^2/1! Limit[FullSimplify[∂z ∂z f[2, z]], z → 0]
N[oλ2]    0.09234573522804657`
oλ3 := (-1)^3/2! Limit[FullSimplify[∂z ∂z ∂z f[3, z]], z → 0]
N[oλ3]    0.20763892055432498`
```

These results are in perfect consistent with known values ( OEIS A074760 , A104539 , A104540 ) .

## 1.2 Li Coefficients from the Hadamard product oμn

The Li coefficients in this section are derived from the Hadamard product.

### Theorem 1.2

Let us define the Li coefficient $_0\mu_n$ using the following two formulas.

$$_0\mu_n = \frac{(-1)^n}{(n-1)!}\left[\frac{d^n}{dz^n}(1-z)^{n-1}\log\,\xi(z)\right]_{z=0} \tag{1.2d}$$

$$\xi(z) = \prod_{k=1}^{\infty}\left(1-\frac{z}{\rho_k}\right) \quad \textit{where, } \rho_k \ (k=1,2,3,\cdots) \textit{ are zeros of } \xi(z) \tag{0.$\rho$}$$

Then, $_0\mu_n$ is expressed as follows:

$$_0\mu_n = \sum_{k=1}^{\infty}\left(1-\left(1-\frac{1}{\rho_k}\right)^n\right) \tag{1.$\mu$}$$

### Proof

**(1) Higher-order Derivatives of $(1-z)^{n-1}$**

This is the same as 1.1 (1) . That is,

$$\frac{d^{n-s}}{dz^{n-s}}(1-z)^{n-1} = (-1)^{n-s}(s)_{n-s}(1-z)^{s-1} \tag{1.1}$$

**(2) Higher-order Derivatives of $log\ \xi(z)$**

Taking the logarithm of both sides of (0.$\rho$) ,

$$log\ \xi(z) = log \prod_{k=1}^{\infty}\left(1-\frac{z}{\rho_k}\right) = \sum_{k=1}^{\infty} log\left(1-\frac{z}{\rho_k}\right)$$

Differentiating both sides with respect to $z$ ,

$$\frac{d}{dz} log\ \xi(z) = \frac{d}{dz}\sum_{k=1}^{\infty} log\left(1-\frac{z}{\rho_k}\right) = \sum_{k=1}^{\infty}\frac{-1/\rho_k}{1-z/\rho_k}$$

i.e.

$$\frac{d^1}{dz^1} log\ \xi(z) = -\sum_{k=1}^{\infty}\frac{1}{\rho_k - z}$$

$$\frac{d^2}{dz^2} log\ \xi(z) = -\sum_{k=1}^{\infty}\frac{1}{(\rho_k - z)^2}$$

$$\frac{d^3}{dz^3} log\ \xi(z) = -\sum_{k=1}^{\infty}\frac{2}{(\rho_k - z)^3}$$

$$\vdots$$

$$\frac{d^s}{dz^s} log\ \xi(z) = -\sum_{k=1}^{\infty}\frac{(s-1)!}{(\rho_k - z)^s} \tag{1.2$\mu$}$$

**(3) Higher-order Derivative Coefficients ( $z=0$ ) of $(1-z)^{n-1} log\ \xi(z)$**

Applying the Leibniz's law to (1.1) and (1.2$\mu$) ,

$$\frac{d^n}{dz^n}(1-z)^{n-1} log\ \xi(z) = -\sum_{s=0}^{n}\binom{n}{s}(-1)^{n-s}(s)_{n-s}(1-z)^{s-1}\sum_{k=1}^{\infty}\frac{(s-1)!}{(\rho_k - z)^s}$$

Since $(0-1)!$ is not possible, changing the initial value of the first $\Sigma$ subscript from $0$ to $1$,

$$\frac{d^n}{dz^n}(1-z)^{n-1} log\ \xi(z) = -\sum_{s=1}^{n}\binom{n}{s}(-1)^{n-s}(s)_{n-s}(1-z)^{s-1}\sum_{k=1}^{\infty}\frac{(s-1)!}{(\rho_k - z)^s}$$

Further,

$$(s)_{n-s} = s(s+1)\cdots(n-1) = \frac{(n-1)!}{(s-1)!}$$

Substituting this for the above,

$$\frac{d^n}{dz^n}(1-z)^{n-1} log\ \xi(z) = -\sum_{s=1}^{n}\binom{n}{s}(-1)^{n-s}\frac{(n-1)!}{(s-1)!}(1-z)^{s-1}\sum_{k=1}^{\infty}\frac{(s-1)!}{(\rho_k - z)^s}$$

i.e.

$$\frac{d^n}{dz^n}(1-z)^{n-1} log\ \xi(z) = -(-1)^n(n-1)!\sum_{k=1}^{\infty}\sum_{s=1}^{n}\binom{n}{s}(1-z)^{s-1}\frac{(-1)^s}{(\rho_k - z)^s}$$

The derivative coefficient of this at $z=0$ , the left edge of the critical strip, is

$$\left[\frac{d^n}{dz^n}(1-z)^{n-1} log\ \xi(z)\right]_{z=0} = -(-1)^n(n-1)!\sum_{k=1}^{\infty}\sum_{s=1}^{n}\binom{n}{s}\frac{(-1)^s}{\rho_k^s}$$

$$= -(-1)^n (n-1)! \sum_{k=1}^{\infty} \left( \sum_{s=0}^{n} \binom{n}{s} \frac{(-1)^s}{\rho_k^{\,s}} - 1 \right)$$

$$= -(-1)^{-n} (n-1)! \sum_{k=1}^{\infty} \left( \left(1 - \frac{1}{\rho_k}\right)^n - 1 \right)$$

i.e.

$$\left[ \frac{d^n}{dz^n} (1-z)^{n-1} \log \xi(z) \right]_{z=0} = (-1)^{-n} (n-1)! \sum_{k=1}^{\infty} \left( 1 - \left( 1 - \frac{1}{\rho_k} \right)^n \right) \tag{1.3$\mu$}$$

**(4) Li Coefficients $_0\mu_n$**

Substituting (1.3$\mu$) for (1.2d) ,

$$_0\mu_n = \sum_{k=1}^{\infty} \left( 1 - \left( 1 - \frac{1}{\rho_k} \right)^n \right) \tag{1.$\mu$}$$

Q.E.D.

## 1.3 $_0\lambda_1$ and $_0\mu_1$

For the Li coefficients $_0\lambda_n$ , $_0\mu_n$ , the following lemma holds especially when $n=1$ .

### Lemma 1.3

Let $\rho_k$ $k=1,2,3,\cdots$ are zeros of the xi function, and let $\rho_{2r-1} = x_r - iy_r$ , $\rho_{2r} = x_r + iy_r$ $r=1,2,3,\cdots$ .

Then,

$$\sum_{r=1}^{\infty} \left( \frac{1}{x_r - iy_r} + \frac{1}{x_r + iy_r} \right) = \frac{(-1)^1}{0!} \lim_{z \to 0} \frac{\xi'(z)}{\xi(z)} = 0.0230957\cdots$$

### Proof

In Theorem 1.2, especially when $n=1$ ,

$$_0\mu_1 = \sum_{k=1}^{\infty} \left( 1 - \left( 1 - \frac{1}{\rho_k} \right)^1 \right) = \sum_{k=1}^{\infty} \frac{1}{\rho_k}$$

If we set $\rho_{2r-1} = x_r - iy_r$ , $\rho_{2r} = x_r + iy_r$ $r=1,2,3,\cdots$ ,

$$_0\mu_1 = \sum_{r=1}^{\infty} \left( \frac{1}{x_r - iy_r} + \frac{1}{x_r + iy_r} \right)$$

On the other hand, in Theorem 1.1 , especially when $n=1$ ,

$$_0\lambda_1 = \frac{(-1)^1}{0!} \lim_{z \to 0} \frac{\xi'(z)}{\xi(z)}$$

Since $_0\lambda_1 = {}_0\mu_1$ ,

$$\sum_{r=1}^{\infty} \left( \frac{1}{x_r - iy_r} + \frac{1}{x_r + iy_r} \right) = \frac{(-1)^1}{0!} \lim_{z \to 0} \frac{\xi'(z)}{\xi(z)} = 0.0230957\cdots$$

Q.E.D.

### Note

According to Theorem 10.1.2 in my paper " **10 Vieta's Formulas on Completed Riemann Zeta** ",

$$\xi(z) = -z(1-z)\pi^{-\frac{z}{2}} \Gamma\left(\frac{z}{2}\right) \zeta(z) = \sum_{r=0}^{\infty} A_r z^r$$

$$A_1 = -\sum_{r_1=1}^{\infty} \frac{2x_{r_1}}{x_{r_1}^2 + y_{r_1}^2} = -\sum_{r=1}^{\infty} \left( \frac{1}{x_r - iy_r} + \frac{1}{x_r + iy_r} \right)$$

On the other hand, According to Theorem 9.1.3 in my paper " **09 Maclaurin Series of Completed Riemann Zeta** ",

$$A_1 = \frac{\log \pi}{2} - \gamma_0 - \frac{1}{2}\psi_0\left(\frac{3}{2}\right) = -0.0230957\cdots$$

From these,

$$\sum_{r=1}^{\infty} \left( \frac{1}{x_r - iy_r} + \frac{1}{x_r + iy_r} \right) = -\frac{\log \pi}{2} + \gamma_0 + \frac{1}{2}\psi_0\left(\frac{3}{2}\right) = 0.0230957\cdots$$

That is,

$$\frac{(-1)^1}{0!} \lim_{z \to 0} \frac{\xi'(z)}{\xi(z)} = -\frac{\log \pi}{2} + \gamma_0 + \frac{1}{2}\psi_0\left(\frac{3}{2}\right)$$

Furthermore,

$$\psi_0\left(\frac{3}{2}\right) = 2 - 2\log 2 - \gamma_0$$

Substituting this for the right side,

$$\frac{(-1)^1}{0!} \lim_{z \to 0} \frac{\xi'(z)}{\xi(z)} = -\frac{\log \pi}{2} + \gamma_0 + \frac{1}{2}\psi_0\left(\frac{3}{2}\right) = 1 - \log 2 - \frac{\log \pi}{2} + \frac{\gamma_0}{2} = 0.0230957\cdots$$

## 2 Li Coefficients at the Right Edge of the Critical Strip

The Li coefficients in this chapter are defined at the right edge of the critical strip $z=1$.

### 2.1 Li Coefficients from the xi function $_1\lambda_n$

The Li coefficients in this section are derived from the Riemann xi function.

**Theorem 2.1**

Let us define the Li coefficient $_1\lambda_n$ using the following two formulas.

$$_1\lambda_n = \frac{1}{(n-1)!}\left[\frac{d^n}{dz^n} z^{n-1} \log \xi(z)\right]_{z=1} \tag{2.1d}$$

$$\xi(z) = -z(1-z)\pi^{-\frac{z}{2}}\Gamma\left(\frac{z}{2}\right)\zeta(z) \tag{0.ξ}$$

Then, $_1\lambda_n$ is expressed as follows:

$$_1\lambda_n = \frac{1}{(n-1)!}\sum_{s=1}^{n}\binom{n}{s}(s)_{n-s}\left[\sum_{t=1}^{s}(-1)^{t-1}(t-1)!\frac{B_{s,t}\left(\xi^{(1)},\xi^{(2)},\ldots,\xi^{(s)}\right)}{\xi(z)^t}\right]_{z=1} \tag{2.λ}$$

Where, $(a)_k$ is the Pochhammer symbol and $B_{n,k}(f_1, f_2, \ldots)$ is the Bell polynomial.

**Proof**

**(1) Higher-order Derivatives of $z^{n-1}$**

$$\frac{d^s}{dz^s} z^{n-1} = \{(n-s)\cdots(n-2)(n-1)\}z^{n-1-s}$$

Using the Pochhammer symbol $(a)_k$ ,

$$\frac{d^s}{dz^s} z^{n-1} = (n-s)_s z^{n-1-s}$$

Replacing $s$ with $n-s$ ,

$$\frac{d^{n-s}}{dz^{n-s}} z^{n-1} = (s)_{n-s} z^{s-1} \tag{2.1}$$

**(2) Higher-order Derivatives of $\log \xi(z)$**

This is the same as 1.1 (2). That is,

$$\frac{d^s}{dz^s}\log \xi(z) = \sum_{t=1}^{s}(-1)^{t-1}(t-1)!\frac{B_{s,t}\left(\xi^{(1)},\xi^{(2)},\ldots,\xi^{(s)}\right)}{\xi(z)^t} \qquad s \ge 1 \tag{1.2λ}$$

**(3) Higher-order Derivatives of $z^{n-1}\log \xi(z)$**

Leibniz's law is

$$\{f(z)g(z)\}^{(n)} = \sum_{s=0}^{n}\binom{n}{s} f^{(n-s)}(z)g^{(s)}(z)$$

Substitute (2.1) and (1.2λ) for this. Then since $n,s \ge 1$,

$$\frac{d^n}{dz^n} z^{n-1}\log \xi(z) = \sum_{s=1}^{n}\binom{n}{s}(s)_{n-s} z^{s-1}\sum_{t=1}^{s}(-1)^{t-1}(t-1)!\frac{B_{s,t}\left(\xi^{(1)},\xi^{(2)},\ldots,\xi^{(s)}\right)}{\xi(z)^t} \tag{2.3λ}$$

**(4) Li Coefficients $_1\lambda_n$**

Substituting (2.3$\lambda$) for (2.1d) ,

$$_1\lambda_n = \frac{1}{(n-1)!}\left[\sum_{s=1}^{n}\binom{n}{s}(s)_{n-s}z^{s-1}\sum_{t=1}^{s}(-1)^{t-1}(t-1)!\,\frac{B_{s,t}\left(\xi^{(1)},\xi^{(2)},\cdots,\xi^{(s)}\right)}{\xi(z)^t}\right]_{z=1}$$

i.e.

$$_1\lambda_n = \frac{1}{(n-1)!}\sum_{s=1}^{n}\binom{n}{s}(s)_{n-s}\left[\sum_{t=1}^{s}(-1)^{t-1}(t-1)!\,\frac{B_{s,t}\left(\xi^{(1)},\xi^{(2)},\cdots,\xi^{(s)}\right)}{\xi(z)^t}\right]_{z=1} \tag{2.$\lambda$}$$

Q.E.D.

The first few are

$$_1\lambda_1 = \frac{1}{0!}\lim_{z\to1}\frac{\xi'(z)}{\xi(z)}$$

$$_1\lambda_2 = \frac{1}{1!}\lim_{z\to1}\left\{\frac{2\xi'(z)}{\xi(z)} - \frac{z\xi'(z)^2}{\xi(z)^2} + \frac{z\xi''(z)}{\xi(z)}\right\}$$

$$_1\lambda_3 = \frac{1}{2!}\lim_{z\to1}\left\{\frac{6\xi'(z)}{\xi(z)} - \frac{6z\xi'(z)^2}{\xi(z)^2} + \frac{2z^2\xi'(z)^3}{\xi(z)^3} + \frac{6z\xi''(z)}{\xi(z)} - \frac{3z^2\xi'(z)\xi''(z)}{\xi(z)^2} + \frac{z^2\xi^{(3)}(z)}{\xi(z)}\right\}$$

$$\vdots$$

Further, when these were calculated using the mathematical processing software *Mathematica*, they were in perfect consistent with the result in 1.1.(4) .

## 2.2 Li Coefficients from the Hadamard product $_1\mu n$

The Li coefficients in this section are derived from the Hadamard product.

### Theorem 2.2

Let us define the Li coefficient $_1\mu_n$ using the following two formulas.

$$_1\mu_n = \frac{1}{(n-1)!}\left[\frac{d^n}{dz^n}z^{n-1}\log\,\xi(z)\right]_{z=1} \tag{2.2d}$$

$$\xi(z) = \prod_{k=1}^{\infty}\left(1-\frac{z}{\rho_k}\right) \quad \textit{where, } \rho_k\ (k=1,2,3,\cdots) \textit{ are zeros of } \xi(z) \tag{0.$\rho$}$$

Then, $_1\mu_n$ is expressed as follows:

$$_1\mu_n = \sum_{k=1}^{\infty}\left(1-\left(1-\frac{1}{\rho_k}\right)^{-n}\right) \tag{2.$\mu$}$$

### Proof

**(1) Higher-order Derivatives of $z^{n-1}$**

This is the same as 2.1 (1) . That is,

$$\frac{d^{n-s}}{dz^{n-s}}z^{n-1} = (s)_{n-s}z^{s-1} \tag{2.1}$$

**(2) Higher-order Derivatives of $log\ \xi(z)$**

This is the same as 1.2 (2) . That is,

$$\frac{d^s}{dz^s} log\ \xi(z) = -\sum_{k=1}^{\infty} \frac{(s-1)!}{(\rho_k - z)^s}$$

However, in this section, we use the following formula, where $\rho_k - z$ is replaced with $z - \rho_k$ .

$$\frac{d^s}{dz^s} log\ \xi(z) = \sum_{k=1}^{\infty} (-1)^{s-1} \frac{(s-1)!}{(z-\rho_k)^s} \tag{2.2$\mu$}$$

**(3) Higher-order Derivative Coefficients ( $z=1$ ) of $z^{n-1} log\ \xi(z)$**

Applying the Leibniz's law to (2.1) and (2.2$\mu$) ,

$$\frac{d^n}{dz^n} z^{n-1} log\ \xi(z) = \sum_{s=0}^{n} \binom{n}{s} (s)_{n-s} z^{s-1} \sum_{k=1}^{\infty} (-1)^{s-1} \frac{(s-1)!}{(z-\rho_k)^s}$$

Since $(0-1)!$ is not possible, changing the initial value of the first $\Sigma$ subscript from $0$ to $1$ ,

$$\frac{d^n}{dz^n} z^{n-1} log\ \xi(z) = \sum_{s=1}^{n} \binom{n}{s} (s)_{n-s} z^{s-1} \sum_{k=1}^{\infty} (-1)^{s-1} \frac{(s-1)!}{(z-\rho_k)^s}$$

Further,

$$(s)_{n-s} = s(s+1)\cdots(n-1) = \frac{(n-1)!}{(s-1)!}$$

Substituting this for the above,

$$\frac{d^n}{dz^n} z^{n-1} log\ \xi(z) = -(n-1)! \sum_{k=1}^{\infty} \sum_{s=1}^{n} \binom{n}{s} z^{s-1} \frac{(-1)^s}{(z-\rho_k)^s}$$

The derivative coefficient of this at $z=1$ , the right edge of the critical strip, is

$$\left[ \frac{d^n}{dz^n} z^{n-1} log\ \xi(z) \right]_{z=1} = -(n-1)! \sum_{k=1}^{\infty} \sum_{s=1}^{n} \binom{n}{s} \frac{(-1)^s}{(1-\rho_k)^s}$$

$$= -(n-1)! \sum_{k=1}^{\infty} \left( \sum_{s=0}^{n} \binom{n}{s} \frac{(-1)^s}{(1-\rho_k)^s} - 1 \right)$$

$$= -(n-1)! \sum_{k=1}^{\infty} \left( \left( 1 - \frac{1}{1-\rho_k} \right)^n - 1 \right)$$

i.e.

$$\left[ \frac{d^n}{dz^n} z^{n-1} log\ \xi(z) \right]_{z=1} = (n-1)! \sum_{k=1}^{\infty} \left( 1 - \left( 1 - \frac{1}{1-\rho_k} \right)^n \right) \tag{2.3$\mu$}$$

**(4) Li Coefficients ${}_1\mu_n$**

Substituting (2.3$\mu$) for (2.2d) ,

$${}_1\mu_n = \sum_{k=1}^{\infty} \left( 1 - \left( 1 - \frac{1}{1-\rho_k} \right)^n \right)$$

Now, for any complex number $\rho_k$ , the following holds:

$$\left( 1 - \frac{1}{1-\rho_k} \right)^n = \left( \frac{-\rho_k}{1-\rho_k} \right)^n = \left( \frac{1-\rho_k}{-\rho_k} \right)^{-n} = \left( \frac{\rho_k - 1}{\rho_k} \right)^{-n} = \left( 1 - \frac{1}{\rho_k} \right)^{-n}$$

Therefore, ${}_1\mu_n$ becomes further as follows.

$$ {}_{1}\mu_{n} = \sum_{k=1}^{\infty} \left( 1 - \left( 1 - \frac{1}{\rho_k} \right)^{-n} \right) \qquad \left( = {}_{0}\mu_{-n} \right) \tag{2.$\mu$} $$

Q.E.D.

## 3 Conditions for Equality of the Li Coefficients at Both Edges

### 3.1 Li Coefficients from the xi function ₀λn & ₁λn

Theorem 1.1 and Theorem 2.1 give the Li coefficients ${}_0\lambda_n$ , ${}_1\lambda_n$ from the xi functions at the left and right edge of the critical strip. The question here is whether ${}_0\lambda_n$ and ${}_1\lambda_n$ are equal. The conclusion is that they are equal. This is shown below as a theorem.

**Theorem 3.1**

Let the Li coefficients ${}_0\lambda_n$ , ${}_1\lambda_n$ at both edges of the critical strip be as follows:

$$ {}_0\lambda_n = \frac{1}{(n-1)!}\sum_{s=1}^{n}\binom{n}{s}(s)_{n-s}(-1)^s\left[\sum_{t=1}^{s}(-1)^{t-1}(t-1)!\,\frac{B_{s,t}\left(\xi^{(1)},\xi^{(2)},\cdots,\xi^{(s)}\right)}{\xi(z)^t}\right]_{z=0} \tag{1.λ}$$

$$ {}_1\lambda_n = \frac{1}{(n-1)!}\sum_{s=1}^{n}\binom{n}{s}(s)_{n-s}\left[\sum_{t=1}^{s}(-1)^{t-1}(t-1)!\,\frac{B_{s,t}\left(\xi^{(1)},\xi^{(2)},\cdots,\xi^{(s)}\right)}{\xi(z)^t}\right]_{z=1} \tag{2.λ}$$

Where,

$$ \xi(z) = -z(1-z)\pi^{-\frac{z}{2}}\Gamma\left(\frac{z}{2}\right)\zeta(z) \tag{0.ξ}$$

Then, the following equation holds.

$$ {}_0\lambda_n = {}_1\lambda_n \qquad n=1,2,3,\cdots \tag{3.1}$$

**Proof**

(1.λ) and (2.λ) can be rewritten as follows:

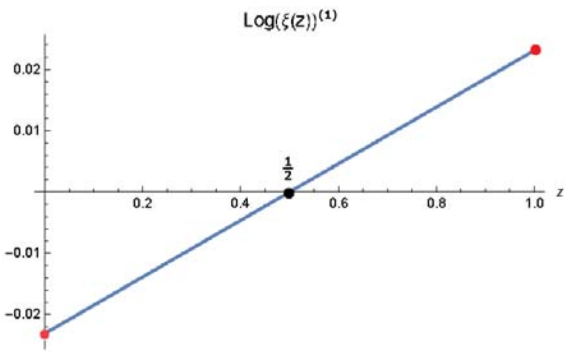
$$ {}_0\lambda_n = \frac{1}{(n-1)!}\sum_{s=1}^{n}\binom{n}{s}(s)_{n-s}(-1)^s\left[\frac{d^s}{dz^s}\log\xi(z)\right]_{z=0}$$

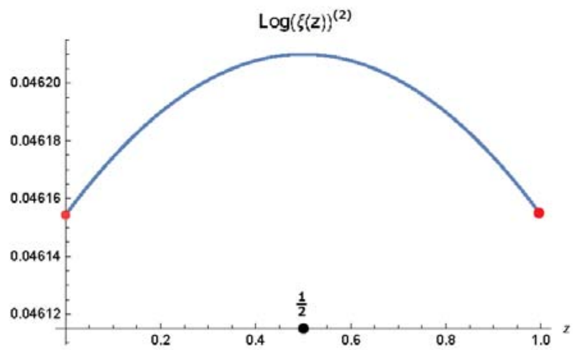
$$ {}_1\lambda_n = \frac{1}{(n-1)!}\sum_{s=1}^{n}\binom{n}{s}(s)_{n-s}\left[\frac{d^s}{dz^s}\log\xi(z)\right]_{z=1}$$

By the functional equation $\xi(z)=\xi(1-z)$ , $\xi(z)$ is line symmetry with respect to $z=1/2$ . So, $\log\xi(z)$ is also line symmetry with respect to $z=1/2$ . And the higher-order derivatives of $\log\xi(z)$ become as follows.

Odd-order derivatives: Point symmetry with respect to $z=1/2$ (example: left figure) .

Even-order derivatives: Line symmetry with respect to $z=1/2$ (example: right figure) .




So, the following equation holds for the higher-order derivative coefficient (red dots) of $\log\xi(z)$ at $s=0$ and $s=1$ .

$$ (-1)^s\left[\frac{d^s}{dz^s}\log\xi(z)\right]_{s=0} = \left[\frac{d^s}{dz^s}\log\xi(z)\right]_{s=1} \qquad for\ s=1,2,3,\cdots$$

Thus, ${}_0\lambda_n = {}_1\lambda_n$ . That is, this holds unconditionally based on the functional equation and the definition.

Q.E.D.

## 3.2 Li Coefficients from the Hadamard product $_0\mu_n$ & $_1\mu_n$

Theorem 1.2 and Theorem 2.2 give the Li coefficients $_0\mu_n$ , $_1\mu_n$ from the Hadamard product at the left and right edge of the critical strip. The question here is whether $_0\mu_n$ and $_1\mu_n$ are equal.

### Lemma 3.2.1

Let $\rho_k$ $k=1,2,3,\cdots$ are zeros of the xi function, and let $\rho_{2r-1}=x_r-iy_r$ , $\rho_{2r}=x_r+iy_r$ $r=1,2,3,\cdots$ . Then,

**(1)** Odd order

$$\sum_{k=1}^{\infty}\left(1-\frac{1}{\rho_k}\right)^{2n-1}=\sum_{r=1}^{\infty}\frac{2}{\left(x_r^2+y_r^2\right)^{2n-1}}\sum_{s=0}^{n-1}(-1)^{n-s-1}\binom{2n-1}{2(n-s-1)}\left(x_r^2+y_r^2-x_r\right)^{2s+1}y_r^{2n-2s-2}$$

$$\sum_{k=1}^{\infty}\left(1-\frac{1}{\rho_k}\right)^{-(2n-1)}=\frac{2}{\left((1-x_r)^2+y_r^2\right)^{2n-1}}\sum_{s=0}^{n-1}(-1)^{n-s-1}\binom{2n-1}{2(n-s-1)}\left(x_r^2+y_r^2-x_r\right)^{2s+1}y_r^{2n-2s-2}$$

**(2)** Even order

$$\sum_{k=1}^{\infty}\left(1-\frac{1}{\rho_k}\right)^{2n}=\sum_{r=1}^{\infty}\frac{2}{\left(x_r^2+y_r^2\right)^{2n}}\sum_{s=0}^{n}(-1)^{n-s}\binom{2n}{2(n-s)}\left(x_r^2+y_r^2-x_r\right)^{2s}y_r^{2n-2s}$$

$$\sum_{k=1}^{\infty}\left(1-\frac{1}{\rho_k}\right)^{-2n}=\sum_{r=1}^{\infty}\frac{2}{\left((1-x_r)^2+y_r^2\right)^{2n}}\sum_{s=0}^{n}(-1)^{n-s}\binom{2n}{2(n-s)}\left(x_r^2+y_r^2-x_r\right)^{2s}y_r^{2n-2s}$$

### Proof

Since the imaginary part of $\rho_k$ has $\pm$, assign $\rho_k$ as in the assumption. Then,

$$\left(1-\frac{1}{\rho_{2r-1}}\right)+\left(1-\frac{1}{\rho_{2r}}\right)=\left(1-\frac{1}{x_r-iy_r}\right)+\left(1-\frac{1}{x_r+iy_r}\right)$$

$$\left(1-\frac{1}{\rho_{2r-1}}\right)^{-1}+\left(1-\frac{1}{\rho_{2r}}\right)^{-1}=\left(1-\frac{1}{x_r-iy_r}\right)^{-1}+\left(1-\frac{1}{x_r+iy_r}\right)^{-1}$$

Making the denominator on the right hand side real,

$$\left(1-\frac{1}{\rho_{2r-1}}\right)+\left(1-\frac{1}{\rho_{2r}}\right)=\left(\frac{x_r^2+y_r^2-x_r}{x_r^2+y_r^2}-i\frac{y_r}{x_r^2+y_r^2}\right)+\left(\frac{x_r^2+y_r^2-x_r}{x_r^2+y_r^2}+i\frac{y_r}{x_r^2+y_r^2}\right)$$

$$\left(1-\frac{1}{\rho_{2r-1}}\right)^{-1}+\left(1-\frac{1}{\rho_{2r}}\right)^{-1}=\left(\frac{x_r^2+y_r^2-x_r}{(1-x_r)^2+y_r^2}-i\frac{y_r}{(1-x_r)^2+y_r^2}\right)$$
$$+\left(\frac{x_r^2+y_r^2-x_r}{(1-x_r)^2+y_r^2}+i\frac{y_r}{(1-x_r)^2+y_r^2}\right)$$

These are complicated, so we will abbreviate them as follows.

$$\frac{x_r^2+y_r^2-x_r}{x_r^2+y_r^2}=A_r\quad,\quad\frac{y_r}{x_r^2+y_r^2}=B_r\quad,\quad\frac{x_r^2+y_r^2-x_r}{(1-x_r)^2+y_r^2}=C_r\ ,\quad\frac{y_r}{(1-x_r)^2+y_r^2}=D_r$$

Then,

When $n=1$

$$\left(1-\frac{1}{\rho_{2r-1}}\right)+\left(1-\frac{1}{\rho_{2r}}\right)=\left(A_r-iB_r\right)+\left(A_r+iB_r\right)=2A_r$$

$$\left(1-\frac{1}{\rho_{2r-1}}\right)^{-1}+\left(1-\frac{1}{\rho_{2r}}\right)^{-1}=\left(C_r-iD_r\right)+\left(C_r+iD_r\right)=2C_r$$

When $n=2$

$$\left(1-\frac{1}{\rho_{2r-1}}\right)^{2}+\left(1-\frac{1}{\rho_{2r}}\right)^{2}=\left(A_r-iB_r\right)^2+\left(A_r+iB_r\right)^2=2\left(A_r^2-B_r^2\right)$$

$$\left(1-\frac{1}{\rho_{2r-1}}\right)^{-2}+\left(1-\frac{1}{\rho_{2r}}\right)^{-2}=\left(C_r-iD_r\right)^2+\left(C_r+iD_r\right)^2=2\left(C_r^2-D_r^2\right)$$

When $n=3$

$$\left(1-\frac{1}{\rho_{2r-1}}\right)^{3}+\left(1-\frac{1}{\rho_{2r}}\right)^{3}=\left(A_r-iB_r\right)^3+\left(A_r+iB_r\right)^3=2\left(A_r^3-3A_rB_r^2\right)$$

$$\left(1-\frac{1}{\rho_{2r-1}}\right)^{-3}+\left(1-\frac{1}{\rho_{2r}}\right)^{-3}=\left(C_r-iD_r\right)^3+\left(C_r+iD_r\right)^3=2\left(C_r^3-3C_rD_r^2\right)$$

When $n=4$

$$\left(1-\frac{1}{\rho_{2r-1}}\right)^{4}+\left(1-\frac{1}{\rho_{2r}}\right)^{4}=\left(A_r-iB_r\right)^4+\left(A_r+iB_r\right)^4=2\left(A_r^4-6A_r^2B_r^2+B_r^4\right)$$

$$\left(1-\frac{1}{\rho_{2r-1}}\right)^{-4}+\left(1-\frac{1}{\rho_{2r}}\right)^{-4}=\left(C_r-iD_r\right)^4+\left(C_r+iD_r\right)^4=2\left(C_r^4-6C_r^2D_r^2+D_r^4\right)$$

The absolute values of the coefficient in 2( ) on the right sides are

$1,\ 1,1,\ 1,3,\ 1,6,1,\ 1,10,5,\ 1,15,15,1,\ \cdots$

This integer sequence matches **OEIS A098158** and is given by the following formula

$$T(n,k)=Binomial(n,2k)\ ,\quad for\ \ n\ge 0\ \ \&\ \ k=0,1,2,\cdots$$

Using this,

Odd order

$$\left(1-\frac{1}{\rho_{2r-1}}\right)^{2n-1}+\left(1-\frac{1}{\rho_{2r}}\right)^{2n-1}=2\sum_{s=0}^{n-1}(-1)^{n-1-s}\binom{2n-1}{2(n-s-1)}A_r^{2s+1}B_r^{2n-2s-2}$$

$$\left(1-\frac{1}{\rho_{2r-1}}\right)^{-(2n-1)}+\left(1-\frac{1}{\rho_{2r}}\right)^{-(2n-1)}=2\sum_{s=0}^{n-1}(-1)^{n-1-s}\binom{2n-1}{2(n-s-1)}C_r^{2s+1}D_r^{2n-2s-2}$$

Even order

$$\left(1-\frac{1}{\rho_{2r-1}}\right)^{2n}+\left(1-\frac{1}{\rho_{2r}}\right)^{2n}=2\sum_{s=0}^{n}(-1)^{n-s}\binom{2n}{2(n-s)}A_r^{2s}B_r^{2n-2s}$$

$$\left(1-\frac{1}{\rho_{2r-1}}\right)^{-2n}+\left(1-\frac{1}{\rho_{2r}}\right)^{-2n}=2\sum_{s=0}^{n}(-1)^{n-s}\binom{2n}{2(n-s)}C_r^{2s}D_r^{2n-2s}$$

Here,

$$\sum_{k=1}^{\infty}\left(1-\frac{1}{\rho_k}\right)^{n}=\sum_{r=1}^{\infty}\left\{\left(1-\frac{1}{\rho_{2r-1}}\right)^{n}+\left(1-\frac{1}{\rho_{2r}}\right)^{n}\right\}$$

$$\sum_{k=1}^{\infty}\left(1-\frac{1}{\rho_k}\right)^{-n}=\sum_{r=1}^{\infty}\left\{\left(1-\frac{1}{\rho_{2r-1}}\right)^{-n}+\left(1-\frac{1}{\rho_{2r}}\right)^{-n}\right\}$$

So, changing $A_r\,, B_r\,, C_r\,, D_r$ back into their original symbols,

Odd order

$$\sum_{k=1}^{\infty}\left(1-\frac{1}{\rho_k}\right)^{2n-1}=2\sum_{r=1}^{\infty}\sum_{s=0}^{n-1}(-1)^{n-s-1}\binom{2n-1}{2(n-s-1)}\left(\frac{x_r^2+y_r^2-x_r}{x_r^2+y_r^2}\right)^{2s+1}\left(\frac{y_r}{x_r^2+y_r^2}\right)^{2n-2s-2}$$

$$\sum_{k=1}^{\infty}\left(1-\frac{1}{\rho_k}\right)^{-(2n-1)}=2\sum_{r=1}^{\infty}\sum_{s=0}^{n-1}(-1)^{n-s-1}\binom{2n-1}{2(n-s-1)}$$

$$\times\left(\frac{x_r^2+y_r^2-x_r}{(1-x_r)^2+y_r^2}\right)^{2s+1}\left(\frac{y_r}{(1-x_r)^2+y_r^2}\right)^{2n-2s-2}$$

i.e.

$$\sum_{k=1}^{\infty}\left(1-\frac{1}{\rho_k}\right)^{2n-1}=\sum_{r=1}^{\infty}\frac{2}{\left(x_r^2+y_r^2\right)^{2n-1}}\sum_{s=0}^{n-1}(-1)^{n-s-1}\binom{2n-1}{2(n-s-1)}\left(x_r^2+y_r^2-x_r\right)^{2s+1}y_r^{2n-2s-2}$$

$$\sum_{k=1}^{\infty}\left(1-\frac{1}{\rho_k}\right)^{-(2n-1)}=\sum_{r=1}^{\infty}\frac{2}{\left((1-x_r)^2+y_r^2\right)^{2n-1}}\sum_{s=0}^{n-1}(-1)^{n-s-1}\binom{2n-1}{2(n-s-1)}\left(x_r^2+y_r^2-x_r\right)^{2s+1}y_r^{2n-2s-2}$$

Even order

$$\sum_{k=1}^{\infty}\left(1-\frac{1}{\rho_k}\right)^{2n}=2\sum_{r=1}^{\infty}\sum_{s=0}^{n}(-1)^{n-s}\binom{2n}{2(n-s)}\left(\frac{x_r^2+y_r^2-x_r}{x_r^2+y_r^2}\right)^{2s}\left(\frac{y_r}{x_r^2+y_r^2}\right)^{2n-2s}$$

$$\sum_{k=1}^{\infty}\left(1-\frac{1}{\rho_k}\right)^{-2n}=2\sum_{r=1}^{\infty}\sum_{s=0}^{n}(-1)^{n-s}\binom{2n}{2(n-s)}\left(\frac{x_r^2+y_r^2-x_r}{(1-x_r)^2+y_r^2}\right)^{2s}\left(\frac{y_r}{(1-x_r)^2+y_r^2}\right)^{2n-2s}$$

i.e.

$$\sum_{k=1}^{\infty}\left(1-\frac{1}{\rho_k}\right)^{2n}=\sum_{r=1}^{\infty}\frac{2}{\left(x_r^2+y_r^2\right)^{2n}}\sum_{s=0}^{n}(-1)^{n-s}\binom{2n}{2(n-s)}\left(x_r^2+y_r^2-x_r\right)^{2s}y_r^{2n-2s}$$

$$\sum_{k=1}^{\infty}\left(1-\frac{1}{\rho_k}\right)^{-2n}=\sum_{r=1}^{\infty}\frac{2}{\left((1-x_r)^2+y_r^2\right)^{2n}}\sum_{s=0}^{n}(-1)^{n-s}\binom{2n}{2(n-s)}\left(x_r^2+y_r^2-x_r\right)^{2s}y_r^{2n-2s}$$

Q.E.D.

Using this lemma, the Li coefficients ${}_0\mu_n$ , ${}_1\mu_n$ can be obtained as follows:

## Theorem 3.2.2

Let the Li coefficients ${}_0\mu_n$ , ${}_1\mu_n$ at both edges of the critical strip be as follows:

$${}_0\mu_n=\sum_{k=1}^{\infty}\left(1-\left(1-\frac{1}{\rho_k}\right)^{n}\right) \qquad (1.\mu)$$

$${}_1\mu_n=\sum_{k=1}^{\infty}\left(1-\left(1-\frac{1}{\rho_k}\right)^{-n}\right) \qquad \left(={}_0\mu_{-n}\right) \qquad (2.\mu)$$

Where,

$$\xi(z)=\prod_{k=1}^{\infty}\left(1-\frac{z}{\rho_k}\right) \qquad \rho_k\ \ (k=1,2,3,\cdots)\ \textit{are zeros of}\ \xi(z) \qquad (0.\rho)$$

Then, when $\rho_{2r-1}=x_r-iy_r$ , $\rho_{2r}=x_r+iy_r$ $r=1,2,3,\cdots$ , the Li coefficients ${}_0\mu_n$ , ${}_1\mu_n$ can be rewritten as follow.

**(1)** Odd order

$${}_0\mu_{2n-1}=2\sum_{r=1}^{\infty}\left\{1-\frac{1}{\left(x_r^2+y_r^2\right)^{2n-1}}\sum_{s=0}^{n-1}(-1)^{n-s-1}\binom{2n-1}{2(n-s-1)}\left(x_r^2+y_r^2-x_r\right)^{2s+1}y_r^{2n-2s-2}\right\}$$

$${}_1\mu_{2n-1}=2\sum_{r=1}^{\infty}\left\{1-\frac{1}{\left((1-x_r)^2+y_r^2\right)^{2n-1}}\sum_{s=0}^{n-1}(-1)^{n-s-1}\binom{2n-1}{2(n-s-1)}\left(x_r^2+y_r^2-x_r\right)^{2s+1}y_r^{2n-2s-2}\right\}$$

**(2)** Even order

$${}_0\mu_{2n}=2\sum_{r=1}^{\infty}\left\{1-\frac{1}{\left(x_r^2+y_r^2\right)^{2n}}\sum_{s=0}^{n}(-1)^{n-s}\binom{2n}{2(n-s)}\left(x_r^2+y_r^2-x_r\right)^{2s}y_r^{2n-2s}\right\}$$

$${}_1\mu_{2n}=2\sum_{r=1}^{\infty}\left\{1-\frac{1}{\left((1-x_r)^2+y_r^2\right)^{2n}}\sum_{s=0}^{n}(-1)^{n-s}\binom{2n}{2(n-s)}\left(x_r^2+y_r^2-x_r\right)^{2s}y_r^{2n-2s}\right\}$$

**Proof**

Li coefficients are

$${}_0\mu_n = \sum_{k=1}^{\infty}\left(1-\left(1-\frac{1}{\rho_k}\right)^n\right) \quad , \quad {}_1\mu_n = \sum_{k=1}^{\infty}\left(1-\left(1-\frac{1}{\rho_k}\right)^{-n}\right)$$

Since,

$$\sum_{k=1}^{\infty}\left(1-\frac{1}{\rho_k}\right)^n = \sum_{r=1}^{\infty}\left\{\left(1-\frac{1}{\rho_{2r-1}}\right)^n + \left(1-\frac{1}{\rho_{2r}}\right)^n\right\}$$

So,

$${}_0\mu_n = \sum_{k=1}^{\infty}\left(1-\left(1-\frac{1}{\rho_k}\right)^n\right) = \sum_{r=1}^{\infty}\left\{2-\left(1-\frac{1}{\rho_{2r-1}}\right)^n - \left(1-\frac{1}{\rho_{2r}}\right)^n\right\}$$

$${}_1\mu_n = \sum_{k=1}^{\infty}\left(1-\left(1-\frac{1}{\rho_k}\right)^{-n}\right) = \sum_{r=1}^{\infty}\left\{2-\left(1-\frac{1}{\rho_{2r-1}}\right)^{-n} - \left(1-\frac{1}{\rho_{2r}}\right)^{-n}\right\}$$

Here, using Lemma 3.2.1 ,

**(1)** Odd order

$${}_0\mu_{2n-1} = 2\sum_{r=1}^{\infty}\left\{1-\frac{1}{\left(x_r^2+y_r^2\right)^{2n-1}}\sum_{s=0}^{n-1}(-1)^{n-s-1}\binom{2n-1}{2(n-s-1)}\left(x_r^2+y_r^2-x_r\right)^{2s+1}y_r^{2n-2s-2}\right\}$$

$${}_1\mu_{2n-1} = 2\sum_{r=1}^{\infty}\left\{1-\frac{1}{\left((1-x_r)^2+y_r^2\right)^{2n-1}}\sum_{s=0}^{n-1}(-1)^{n-s-1}\binom{2n-1}{2(n-s-1)}\left(x_r^2+y_r^2-x_r\right)^{2s+1}y_r^{2n-2s-2}\right\}$$

**(2)** Even order

$${}_0\mu_{2n} = 2\sum_{r=1}^{\infty}\left\{1-\frac{1}{\left(x_r^2+y_r^2\right)^{2n}}\sum_{s=0}^{n}(-1)^{n-s}\binom{2n}{2(n-s)}\left(x_r^2+y_r^2-x_r\right)^{2s}y_r^{2n-2s}\right\}$$

$${}_1\mu_{2n} = 2\sum_{r=1}^{\infty}\left\{1-\frac{1}{\left((1-x_r)^2+y_r^2\right)^{2n}}\sum_{s=0}^{n}(-1)^{n-s}\binom{2n}{2(n-s)}\left(x_r^2+y_r^2-x_r\right)^{2s}y_r^{2n-2s}\right\}$$

Q.E.D.

**Note**

Observing these, we can see that for both odd and even, the only difference between the Li coefficients at both edges of the critical strip are $x_r^2$ and $(1-x_r)^2$, and the difference between them is $1-2x_r$.

## 3.3 Li Coefficients when $Re(\rho_k) = 1/2$

When the zeros of the xi function are on the critical line, we obtain the following theorem from Theorem 3.2.2 and Theorem 3.1 .

**Theorem 3.3**

Let the Li coefficients ${}_0\mu_n$ , ${}_1\mu_n$ at both edges of the critical strip be as follows:

$${}_0\mu_n = \sum_{k=1}^{\infty}\left(1-\left(1-\frac{1}{\rho_k}\right)^n\right) \tag{1.$\mu$}$$

$${}_1\mu_n = \sum_{k=1}^{\infty}\left(1-\left(1-\frac{1}{\rho_k}\right)^{-n}\right) \qquad \left(={}_0\mu_{-n}\right) \tag{2.$\mu$}$$

Where,

$$\xi(z) = \prod_{k=1}^{\infty}\left(1-\frac{z}{\rho_k}\right) \qquad \rho_k \ \ (k=1,2,3,\cdots) \text{ are zeros of } \xi(z) \tag{0.$\rho$}$$

Then, if $Re(\rho_k) = 1/2 \quad k=1,2,3,\cdots$ , the followings hold.

**(1)** Odd order

$${}_0\mu_{2n-1} = {}_1\mu_{2n-1}$$

$$= 2\sum_{r=1}^{\infty}\left\{1 - \frac{1}{\left(y_r^2+1/4\right)^{2n-1}}\sum_{s=0}^{n-1}(-1)^{n-s-1}\binom{2n-1}{2(n-s-1)}\left(y_r^2-1/4\right)^{2s+1}y_r^{2n-2s-2}\right\} \tag{3.1}$$

**(2)** Even order

$${}_0\mu_{2n} = {}_1\mu_{2n}$$

$$= 2\sum_{r=1}^{\infty}\left\{1 - \frac{1}{\left(y_r^2+1/4\right)^{2n}}\sum_{s=0}^{n}(-1)^{n-s}\binom{2n}{2(n-s)}\left(y_r^2-1/4\right)^{2s}y_r^{2n-2s}\right\} \tag{3.2}$$

**(3)** $${}_0\lambda_n = {}_1\lambda_n = {}_0\mu_n = {}_1\mu_n \qquad n=1,2,3,\cdots \tag{3.3}$$

### Proof

Putting $\rho_{2r-1} = 1/2 - iy_r\,,\ \rho_{2r} = 1/2 + iy_r \quad r=1,2,3,\cdots$ in Theorem 3.2.2 , we obtain (3.1) and (3.2) .
Next, since ${}_0\lambda_n = {}_0\mu_n$ in the Chapter 1 and ${}_0\lambda_n = {}_1\lambda_n$ from Theorem 3.1 , we obtain (3.3)

Q.E.D.

**cf.**

According to Theorem 15.4.3 in my paper " **15 Li Coefficients on the Critical Line** ", when $Re(\rho_k) = 1/2 \quad k=1,2,3,\cdots$

$${}_0\mu_{2n-1} = \sum_{r=1}^{\infty}\frac{1}{4^{2n-2}\left(\left(1/4+y_r^2\right)^{(2n-1)/2}\right)^2}\left(\sum_{s=0}^{n-1}(-1)^s 4^s\binom{2n-1}{2s}y_r^{2s}\right)^2 \qquad n=1,2,3,\cdots$$

$${}_0\mu_{2n} = \sum_{r=1}^{\infty}\frac{y^2}{4^{2n-2}\left(1/4+y_r^2\right)^{2n}}\left(\sum_{s=0}^{n-1}(-1)^s 4^s\binom{2n}{2s+1}y_r^{2s}\right)^2 \qquad n=1,2,3,\cdots$$

Since these are sums of squares of real numbers, they satisfy Li's Criterion. In fact, these are equivalent to (3.1) and (3.2) .
For example,

$${}_0\mu_3 = 2\sum_{r=1}^{\infty}\left\{1 - \frac{-3y_r^2\left(y_r^2-1/4\right) + \left(y_r^2-1/4\right)^2}{\left(y_r^2+1/4\right)^3}\right\} = \sum_{r=1}^{\infty}\frac{1-24y_r^2+144y_r^4}{4^2\left(y_r^2+1/4\right)^3} = \sum_{r=1}^{\infty}\frac{\left(1-12y_r^2\right)^2}{4^2\left(y_r^2+1/4\right)^3}$$

$${}_0\mu_4 = 2\sum_{r=1}^{\infty}\left\{1 - \frac{y_r^4-6y_r^2\left(y_r^2-1/4\right)^2+\left(y_r^2-1/4\right)^4}{\left(y_r^2+1/4\right)^4}\right\} = \sum_{r=1}^{\infty}\frac{y_r^2-8y_r^4+16y_r^6}{\left(y_r^2+1/4\right)^4} = \sum_{r=1}^{\infty}\frac{y_r^2\left(4-16y_r^2\right)^2}{4^2\left(y_r^2+1/4\right)^4}$$

So, (3.1) and (3.2) also satisfy Li's Criterion.

### Note

**1.** $Re(\rho_k) = 1/2$ is a sufficient condition for ${}_0\mu_n = {}_1\mu_n$ .

**2.** $Re(\rho_k) = 1/2$ is a necessary condition likely for ${}_0\mu_n = {}_1\mu_n$ .

If not so, we must solve the following system of equations.

$${}_0\mu_{2n-1} = {}_1\mu_{2n-1} \quad , \quad {}_0\mu_{2n} = {}_1\mu_{2n} \qquad n=1,2,3,\cdots$$

This is a system of equations where both the number of unknowns and the number of equations are uncertain.
So, it is difficult to imagine that there could be any solution other than an identity

**3.** However, for the purposes of this chapter, sufficient condition alone is sufficient.

### Confirmation Calculation

Let us calculate these using the zeros $1/2 \pm iy_r \quad r=1,2,3,\cdots,100{,}000$ on the critical line using mathematical processing software *Mathematica*. The result is as follows.

$y_{r\_} := \mathrm{Im}[\mathrm{ZetaZero}[r]]$

**2n-1**

$$o\mu_{n\_}[m\_] := 2\sum_{r=1}^{m}\left(1 - \frac{1}{\left(y_r^2 + 1/4\right)^{2n-1}}\sum_{s=0}^{n-1}(-1)^{n-s-1}\,\mathrm{Binomial}[2n-1, 2(n-s-1)]\left(y_r^2 - 1/4\right)^{2s+1} y_r^{2n-2s-2}\right)$$

**2n**

$$e\mu_{n\_}[m\_] := 2\sum_{r=1}^{m}\left(1 - \frac{1}{\left(y_r^2 + 1/4\right)^{2n}}\sum_{s=0}^{n}(-1)^{n-s}\,\mathrm{Binomial}[2n, 2(n-s)]\left(y_r^2 - 1/4\right)^{2s} y_r^{2n-2s}\right)$$

$N[\{o\mu_1[100000], e\mu_1[100000], o\mu_2[100000], e\mu_2[100000]\}]$

$\{0.0230736, 0.0922575, 0.20744, 0.368437\}$

These results are almost the same as the 1.1 calculation . For reference, this calculation took 16 hours and 20 minutes on my computer ( Intel Core i7-9750H , 16GB ) .

## 4 Proof of the Riemann Hypothesis

**Theorem 4.1**

When $\rho_k \quad k=1,2,3,\cdots$ are non-trivial zeros of the Riemann zeta function $\zeta(z)$ ,

$Re(\rho_k)=1/2 \quad k=1,2,3,\cdots$ .

**Proof**

First, the nontrivial zeros of the Riemann zeta function $\zeta(z)$ are the same as the zeros of the xi function.

**1.** When $\rho_k \quad k=1,2,3,\cdots$ are zeros of xi function and $\rho_{2r-1}=1/2-iy_r$ , $\rho_{2r}=1/2+iy_r \quad r=1,2,3,\cdots$ ,

From Theorems 3.3 , the followings hold

$$ {}_0\mu_{2n-1} = {}_1\mu_{2n-1} $$
$$ = 2\sum_{r=1}^{\infty}\left\{1-\frac{1}{\left(y_r^2+1/4\right)^{2n-1}}\sum_{s=0}^{n-1}(-1)^{n-s-1}\binom{2n-1}{2(n-s-1)}\left(y_r^2-1/4\right)^{2s+1}y_r^{2n-2s-2}\right\} \tag{3.1} $$

$$ {}_0\mu_{2n} = {}_1\mu_{2n} $$
$$ = 2\sum_{r=1}^{\infty}\left\{1-\frac{1}{\left(y_r^2+1/4\right)^{2n}}\sum_{s=0}^{n}(-1)^{n-s}\binom{2n}{2(n-s)}\left(y_r^2-1/4\right)^{2s}y_r^{2n-2s}\right\} \tag{3.2} $$

$$ {}_0\lambda_n = {}_1\lambda_n = {}_0\mu_n = {}_1\mu_n \qquad n=1,2,3,\cdots \tag{3.3} $$

**2.** Especially when $n=1$ ,

$$ {}_0\mu_1 = {}_1\mu_1 = 2\sum_{r=1}^{\infty}\left\{1-\frac{y_r^2-1/4}{y_r^2+1/4}\right\} = \sum_{r=1}^{\infty}\frac{1}{y_r^2+1/4} = \sum_{r=1}^{\infty}\left(\frac{1}{1/2-iy_r}+\frac{1}{1/2+iy_r}\right) $$

On the other hand, from Theorem 1.1 ,

$$ {}_0\lambda_1 = \frac{(-1)^1}{0!}\lim_{z\to 0}\frac{\xi'(z)}{\xi(z)} = 0.0230957\cdots $$

From (3.3) ,

$$ \sum_{r=1}^{\infty}\left(\frac{1}{1/2-iy_r}+\frac{1}{1/2+iy_r}\right) = \frac{(-1)^1}{0!}\lim_{z\to 0}\frac{\xi'(z)}{\xi(z)} = 0.0230957\cdots \tag{4.1} $$

**3.** Now, suppose that in addition to the zeros on the critical line, there are also zeros off the critical line.

It is known that a set of such zeros should consist of the following four.

$$ 1/2-\alpha_s \pm i\beta_s \ , \ 1/2+\alpha_s \pm i\beta_s \quad \left(0<\alpha_s<1/2 \ , \ \beta_s>0\right) $$

Then, from Lemma 1.3 , the following equation has to hold .

$$ \sum_{r=1}^{\infty}\left(\frac{1}{1/2-iy_r}+\frac{1}{1/2+iy_r}\right)+\sum_{s=1}\left(\frac{1}{1/2-\alpha_s-i\beta_s}+\frac{1}{1/2-\alpha_s+i\beta_s}\right) $$
$$ +\sum_{s=1}\left(\frac{1}{1/2+\alpha_s-i\beta_s}+\frac{1}{1/2+\alpha_s+i\beta_s}\right) = \frac{(-1)^1}{0!}\lim_{z\to 0}\frac{\xi'(z)}{\xi(z)} $$
$$ = 0.0230957\cdots $$

i.e.

$$ \sum_{r=1}^{\infty}\left(\frac{1}{1/2-iy_r}+\frac{1}{1/2+iy_r}\right)+\sum_{s=1}\left\{\frac{1-2\alpha_s}{(1/2-\alpha_s)^2+\beta_s^2}+\frac{1+2\alpha_s}{(1/2+\alpha_s)^2+\beta_s^2}\right\} = \frac{(-1)^1}{0!}\lim_{z\to 0}\frac{\xi'(z)}{\xi(z)} $$
$$ = 0.0230957\cdots \tag{4.2} $$

However, since $0 < 1-2\alpha_s < 1+2\alpha_s < 2$ for $0 < \alpha_s < 1/2$,

$$\sum_{s=1} \left\{ \frac{1-2\alpha_s}{(1/2-\alpha_s)^2+\beta_s^2} + \frac{1+2\alpha_s}{(1/2+\alpha_s)^2+\beta_s^2} \right\} > 0$$

Therefore, (4.2) contradicts (4.1). Thus, there should be no zeros outside the critical line within the critical strip.

Q.E.D.

Thus, the Riemann hypothesis is established as a theorem.

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**Alien's Mathematics**

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