Summary of Dirichlet Beta Function

1 Dirichlet Beta Generating Functions

sech x , sec x and csc x can be expanded to Fourier series and Taylor series. And if the termwise higher order integration of these is carried out, Dirichlet Beta at a natural number are obtained.

Where, these are automorphisms which are expressed by lower betas. However, in this chapter, we stop those so far. The work that obtain the non-automorphism formulas by removing lower betas from these is done in the next chapter **" 2 Formulas for Dirichlet Beta " .**

In this chapter, we obtain the following polynomials from the beta generating functions of each family of sech, sec and csc . Where, Dirichlet Beta and Dirichlet Lambda are as follows.

$$\beta(x)=\sum_{r=0}^{\infty}\frac{(-1)^r}{(2r+1)^x}\ ,\ \lambda(x)=\sum_{r=1}^{\infty}\frac{1}{(2r-1)^x}$$

Bernoulli numbers and Euler numbers are as follows.

$$B_0=1,\ B_2=1/6,\ B_4=-1/30,\ B_6=1/42,\ B_8=-1/30,\cdots$$
$$E_0=1,\ E_2=-1,\ E_4=5,\ E_6=-61,\ E_8=1385,\ \cdots$$

Harmonic number is $H_s=\sum_{t=1}^{s}1/t=\psi(1+s)+\gamma$

$$\beta(n)=\sum_{r=0}^{\infty}\frac{(-1)^r e^{-(2r+1)x}}{(2r+1)^n}-\frac{(-1)^n}{2}\sum_{r=0}^{\infty}\frac{E_{2r}x^{2r+n}}{(2r+n)!}-\sum_{s=1}^{n-1}\frac{(-1)^s x^s}{s!}\beta(n-s)$$

$$\sum_{r=0}^{\infty}\frac{(-1)^r \sin\{(2r+1)x\}}{(2r+1)^{2n+1}}-\frac{(-1)^n}{2}\sum_{r=0}^{\infty}\frac{|E_{2r}|x^{2n+1+2r}}{(2n+1+2r)!}=\sum_{s=0}^{n-1}\frac{(-1)^s x^{2s+1}}{(2s+1)!}\beta(2n-2s)$$

$$\sum_{r=0}^{\infty}\frac{(-1)^r \cos\{(2r+1)x\}}{(2r+1)^{2n}}-\frac{(-1)^n}{2}\sum_{r=0}^{\infty}\frac{|E_{2r}|x^{2n+2r}}{(2n+2r)!}=\sum_{s=0}^{n-1}\frac{(-1)^s x^{2s}}{(2s)!}\beta(2n-2s)$$

$$\sum_{r=0}^{\infty}\frac{(-1)^r \cos\{(2r+1)x\}}{(2r+1)^{2n+1}}=\sum_{s=0}^{n}\frac{(-1)^s x^{2s}}{(2s)!}\beta(2n+1-2s)$$

$$\sum_{r=0}^{\infty}\frac{(-1)^r \sin\{(2r+1)x\}}{(2r+1)^{2n}}=\sum_{s=0}^{n-1}\frac{(-1)^s x^{2s+1}}{(2s+1)!}\beta(2n-1-2s)$$

$$\begin{aligned}\beta(2n)=&\frac{(-1)^n}{2(2n-1)!}\left(\frac{\pi}{2}\right)^{2n-1}\left(\log\frac{\pi}{4}-H_{2n-1}\right)\\&+\frac{(-1)^n}{2}\sum_{r=1}^{\infty}\frac{\left(2^{2r}-2\right)|B_{2r}|}{2r(2r+2n-1)!}\left(\frac{\pi}{2}\right)^{2n-1+2r}\\&-\sum_{s=1}^{n-1}\frac{(-1)^s}{(2s-1)!}\left(\frac{\pi}{2}\right)^{2s-1}\lambda(2n+1-2s)\end{aligned}$$

Furthermore, if the termwise higher order differentiation of the Fourier series of each family of sech and sec are carried out, the following expressions are obtained.

$$\beta(-n)=\frac{1}{2^{n+1}}\sum_{r=0}^{n}(-1)^r\,{}_nK_r \qquad n=1,2,3,\cdots$$

$$\beta(-2n)=\frac{1}{2^{2n+1}}\sum_{r=0}^{2n}(-1)^r\,{}_{2n}K_r \qquad n=1,2,3,\cdots$$

$$=\frac{E_{2n}}{2n} \qquad n=1,2,3,\cdots$$

Where, ${}_nK_r$ is a kind of Eulderian Number and is defined as follows.

$${}_nK_r=\sum_{k=0}^{r}(-1)^k\binom{n+1}{k}(2r+1-2k)^n \qquad n=1,2,3,\cdots$$

2 Formulas for Dirichlet Beta

Here, removing the lower betas from the the automorphism formulas in the previous chapter, we obtain the following non-automorphism formulas. Where, Bernoulli numbers and Euler numbers are as follows.

$$B_0=1,\ B_2=1/6,\ B_4=-1/30,\ B_6=1/42,\ B_8=-1/30,\cdots$$

$$E_0=1,\ E_2=-1,\ E_4=5,\quad E_6=-61,\ E_8=1385,\ \cdots$$

And, gamma function and incomplete gamma function were as follows.

$$\Gamma(p)=\int_0^\infty t^{p-1}e^{-t}dt \quad , \quad \Gamma(p,x)=\int_x^\infty t^{p-1}e^{-t}dt$$

2.1 Formulas for Beta at natural number

For $0<x\le\pi/2$,

$$\beta(n)=\sum_{r=0}^{\infty}\sum_{s=0}^{n-1}\frac{(2r+1)^s x^s}{s!}\frac{(-1)^r e^{-(2r+1)x}}{(2r+1)^n}+\frac{x^n}{2}\sum_{r=0}^{\infty}\binom{-n}{2r}\frac{E_{2r}x^{2r}}{(n+2r)!}$$

Especially,

$$\beta(n)=\sum_{r=0}^{\infty}\sum_{s=0}^{n-1}\frac{(2r+1)^s}{s!\,2^s}\frac{(-1)^r e^{-(r+1/2)}}{(2r+1)^n}+\frac{1}{2^{n+1}}\sum_{r=0}^{\infty}\binom{-n}{2r}\frac{E_{2r}}{(n+2r)!\,2^{2r}}$$

$$\beta(n)=\sum_{r=0}^{\infty}\sum_{s=0}^{n-1}\frac{(2r+1)^s}{s!}\frac{(-1)^r e^{-(2r+1)}}{(2r+1)^n}+\frac{1}{2}\sum_{r=0}^{\infty}\binom{-n}{2r}\frac{E_{2r}}{(n+2r)!}$$

Example

$$\beta(4)=\sum_{r=0}^{\infty}\left\{1+\frac{2r+1}{1!\,2^1}+\frac{(2r+1)^2}{2!\,2^2}+\frac{(2r+1)^3}{3!\,2^3}\right\}\frac{(-1)^r e^{-r-\frac{1}{2}}}{(2r+1)^4}+\frac{1}{2^5}\sum_{r=0}^{\infty}\binom{-4}{2r}\frac{E_{2r}}{(4+2r)!\,2^{2r}}$$

$$\beta(4)=\sum_{r=0}^{\infty}\left\{1+\frac{2r+1}{1!}+\frac{(2r+1)^2}{2!}+\frac{(2r+1)^3}{3!}\right\}\frac{(-1)^r e^{-2r-1}}{(2r+1)^4}+\frac{1}{2}\sum_{r=0}^{\infty}\binom{-4}{2r}\frac{E_{2r}}{(4+2r)!}$$

2.2 Formulas for Beta at even number

For $0<x\le\pi/2$,

$$\beta(2n)=\sum_{r=0}^{\infty}\sum_{s=0}^{n-1}\frac{|E_{2s}|\{(2r+1)x\}^{2s}}{(2s)!}\frac{(-1)^r\cos\{(2r+1)x\}}{(2r+1)^{2n}}$$
$$-\frac{(-1)^n}{2}\sum_{r=0}^{\infty}\left\{\sum_{s=0}^{n-1}\binom{2n+2r}{2s}E_{2s}\right\}\frac{|E_{2r}|x^{2n+2r}}{(2n+2r)!}$$

$$\beta(2n)=-\frac{1}{x}\sum_{r=1}^{\infty}\sum_{s=0}^{n-1}\frac{(-1)^s B_{2s}(2^{2s}-2)\{(2r+1)x\}^{2s}}{(2s)!}\frac{(-1)^r\sin\{(2r+1)x\}}{(2r+1)^{2n+1}}$$
$$+(-1)^n\frac{x^{2n}}{2}\sum_{r=1}^{\infty}\left\{\sum_{s=0}^{n-1}\frac{(2^{2s}-2)B_{2s}}{(2s)!\,(2n+1+2r-2s)!}\right\}\frac{|E_{2r}|x^{2r}}{2r}$$

Especially,

$$\beta(2n)=\frac{(-1)^{n-1}}{2}\sum_{r=0}^{\infty}\left\{\sum_{s=0}^{n-1}\binom{2n+2r}{2s}E_{2s}\right\}\frac{|E_{2r}|}{(2n+2r)!}\left(\frac{\pi}{2}\right)^{2n+2r}$$

Example

$$\beta(4)=-\frac{1}{2}\sum_{r=0}^{\infty}\left\{\binom{4+2r}{0}E_0+\binom{4+2r}{2}E_2\right\}\frac{|E_{2r}|}{(4+2r)!}\left(\frac{\pi}{2}\right)^{4+2r}$$

$$\beta(6)=\frac{1}{2}\sum_{r=0}^{\infty}\left\{\binom{6+2r}{0}E_0+\binom{6+2r}{2}E_2+\binom{6+2r}{4}E_4\right\}\frac{|E_{2r}|}{(6+2r)!}\left(\frac{\pi}{2}\right)^{6+2r}$$

2.3 Formulas for Beta at odd number

For $0<x\leq\pi/2$,

$$\beta(2n-1) = \sum_{r=0}^{\infty}\sum_{s=0}^{n-1}\frac{|E_{2s}|\{(2r+1)x\}^{2s}}{(2s)!}\frac{(-1)^r\cos\{(2r+1)x\}}{(2r+1)^{2n-1}}$$

$$\beta(2n-1) = -\frac{1}{x}\sum_{r=0}^{\infty}\sum_{s=0}^{n-1}\frac{(-1)^s B_{2s}\left(2^{2s}-2\right)\{(2r+1)x\}^{2s}}{(2s)!}\frac{(-1)^r\sin\{(2r+1)x\}}{(2r+1)^{2n}}$$

Especially,

$$\beta(2n-1) = \frac{\pi}{4}\frac{|E_{2n-2}|}{(2n-2)!}\left(\frac{\pi}{2}\right)^{2n-2}$$

2.4 Formulas for Beta at complex number

When p is a complex number such that $p\neq 1,0,-1,-2,\cdots$,

For $x=u+vi$ s.t. $0<|x|\leqq 2\pi$, $u\geqq 0$,

$$\beta(p) = \sum_{r=0}^{\infty}\frac{\Gamma\{p,(2r+1)x\}}{\Gamma(p)}\frac{(-1)^r}{(2r+1)^p}+\frac{x^p}{2}\sum_{r=0}^{\infty}\binom{-p}{2r}\frac{E_{2r}x^{2r}}{\Gamma(p+1+2r)}$$

Especially,

$$\beta(p) = \sum_{r=0}^{\infty}\frac{\Gamma(p,2r+1)}{\Gamma(p)}\frac{(-1)^r}{(2r+1)^p}+\frac{1}{2}\sum_{r=0}^{\infty}\binom{-p}{2r}\frac{E_{2r}}{\Gamma(p+1+2r)}$$

3 Global definition of Dirichlet Beta and Generalized Euler Number

Diriclet beta function is defined on the whole complex plane with patches as follows.

$$\beta(p) = \begin{cases}\displaystyle\sum_{r=1}^{\infty}\frac{(-1)^{r-1}}{(2r-1)^p} & Re(p)\geqq 0\\ \displaystyle\left(\frac{2}{\pi}\right)^{1-p}\cos\frac{p\pi}{2}\Gamma(1-p)\sum_{r=1}^{\infty}\frac{(-1)^{r-1}}{(2r-1)^{1-p}} & Re(p)<0\end{cases}$$

This is inconvenient. so, we focus on the following sequence.

$${}_nB_r = \sum_{s=0}^{r}(-1)^{r-s}{}_rC_s\left(s-\frac{1}{2}\right)^n \qquad r=0,1,2,\cdots,n$$

Using this sequence, we can define Diriclet beta function on the whole complex plane as follows.

Definition 3.2.1

$$\beta(p) = \sum_{r=1}^{\infty}\frac{1}{2^{r+1}}\sum_{s=1}^{r}(-1)^{s-1}\binom{r}{s}(2s-1)^{-p}$$

Furthermore, by using this sequence, Euler Number can be defined on the whole complex plane.

Definition 3.3.1

$$E_p = \sum_{r=1}^{\infty}\frac{1}{2^r}\sum_{s=1}^{r}(-1)^{s-1}\binom{r}{s}(2s-1)^p$$

4 Completed Dirichlet Beta

In **4.1**, symmetric functional equations are derived from functional equations.

Formula 4.1.1

$$\left(\frac{2}{\sqrt{\pi}}\right)^{1+z}\Gamma\left(\frac{1}{2}+\frac{z}{2}\right)\beta(z) = \left(\frac{2}{\sqrt{\pi}}\right)^{2-z}\Gamma\left(\frac{1}{2}+\frac{1-z}{2}\right)\beta(1-z)$$

$$\left(\frac{2}{\sqrt{\pi}}\right)^{\frac{3}{2}+z}\Gamma\left(\frac{3}{4}+\frac{z}{2}\right)\beta\left(\frac{1}{2}+z\right) = \left(\frac{2}{\sqrt{\pi}}\right)^{\frac{3}{2}-z}\Gamma\left(\frac{3}{4}-\frac{z}{2}\right)\beta\left(\frac{1}{2}-z\right)$$

In **4.2**, we define the completed Dirichlet beta functions $\omega(z)$, $\Omega(z)$ as follows, respectively.

$$\omega(z) = \left(\frac{2}{\sqrt{\pi}}\right)^{1+z}\Gamma\left(\frac{1+z}{2}\right)\beta(z)$$

$$\Omega(z) = \left(\frac{2}{\sqrt{\pi}}\right)^{\frac{3}{2}+z}\Gamma\left\{\frac{1}{2}\left(\frac{3}{2}+z\right)\right\}\beta\left(\frac{1}{2}+z\right)$$

Then, Formula 4.1.1 is expressed as follows.

$$\omega(z) = \omega(1-z)$$

$$\Omega(z) = \Omega(-z)$$

From the latter, we can see that $\Omega(z)$ is an even function. Therefore, $\Omega(z)$ has the same properties as completed Riemann zeta function $\Xi(z)$. (See " 07 Completed Riemann Zeta ".) And, as in $\Xi(z)$, the following theorem holds.

Theorem 4.2.1

If Dirichlet beta function $\beta(z)$ has a non-trivial zero whose real part is not $1/2$, the one set consists of the following four.

$$1/2+\alpha_1 \pm i\delta_1 \quad , \quad 1/2-\alpha_1 \pm i\delta_1 \qquad (0 < \alpha_1 < 1/2)$$

05 Factorization of Completed Dirichlet Beta

In **5.1**, the following Hadamard product is derived.

Formula 5.1.1

Let completed beta function be as follows.

$$\omega(z) = \left(\frac{2}{\sqrt{\pi}}\right)^{1+z}\Gamma\left(\frac{1+z}{2}\right)\beta(z)$$

When non-trivial zeros of $\beta(z)$ are $z_k = x_k \pm iy_k \quad k=1,2,3,\cdots$ and γ is Euler-Mascheroni constant, $\omega(z)$ is expressed by the Hadamard product as follows.

$$\omega(z) = e^{\left(\frac{3\log\pi}{2}-\frac{\gamma}{2}-\log 2-4\log\Gamma\left(\frac{3}{4}\right)\right)z}\prod_{k=1}^{\infty}\left(1-\frac{z}{z_k}\right)e^{\frac{z}{z_k}}$$

$$\omega(z) = e^{\left(\frac{3\log\pi}{2}-\frac{\gamma}{2}-\log 2-4\log\Gamma\left(\frac{3}{4}\right)\right)z}\prod_{n=1}^{\infty}\left(1-\frac{2x_n z}{x_n^2+y_n^2}+\frac{z^2}{x_n^2+y_n^2}\right)e^{\frac{2x_n z}{x_n^2+y_n^2}}$$

And, the following special values are obtained.

$$\prod_{n=1}^{\infty}\left(1-\frac{2x_n-1}{x_n^2+y_n^2}\right)e^{\frac{2x_n}{x_n^2+y_n^2}} = e^{4\log\Gamma\left(\frac{3}{4}\right)+\frac{\gamma}{2}+\log 2-\frac{3\log\pi}{2}} = 1.08088915\cdots$$

$$\prod_{n=1}^{\infty}\left\{1-\frac{1}{(x_n+iy_n)^2}\right\}\left\{1-\frac{1}{(x_n-iy_n)^2}\right\} = \omega(-1) = 1.16624361\cdots$$

In **5.2**, we consider how the formulas in the previous section are expressed when non-trivial zeros whose real part is 1/2 and non-trivial zeros whose real part is not 1/2 are mixed. Then, we obtain the following theorems.

Theorem 5.2.2

Let γ be Euler-Mascheroni constant, non-trivial zeros of Dirichlet beta function are x_n+iy_n $n=1,2,3,\cdots$. Among them, zeros whose real part is 1/2 are $1/2\pm iy_r$ $r=1,2,3,\cdots$ and zeros whose real parts is not 1/2 are $1/2\pm\alpha_s\pm i\delta_s$ $(0<\alpha_s<1/2)$ $s=1,2,3,\cdots$. Then the following expressions hold.

$$\prod_{n=1}^{\infty}\left(1-\frac{2x_n-1}{x_n^2+y_n^2}\right) = 1$$

$$\sum_{n=1}^{\infty}\frac{2x_n}{x_n^2+y_n^2} = \sum_{r=1}^{\infty}\frac{1}{1/4+y_r^2}+\sum_{s=1}\left\{\frac{1+2\alpha_s}{(1/2+\alpha_s)^2+\delta_s^2}+\frac{1-2\alpha_s}{(1/2-\alpha_s)^2+\delta_s^2}\right\}$$

$$\sum_{n=1}^{\infty}\frac{2x_n}{x_n^2+y_n^2} = 4\log\Gamma\left(\frac{3}{4}\right)+\frac{\gamma}{2}+\log 2-\frac{3\log\pi}{2} = 0.07778398\cdots$$

Formula 5.2.3 (Special values)

When non-trivial zeros of Dirichlet beta function are $x_k\pm iy_k$ $k=1,2,3,\cdots$, the following expressions hold.

$$\prod_{n=1}^{\infty}\left(1-\frac{1}{x_n+iy_n}\right)\left(1-\frac{1}{x_n-iy_n}\right) = 1$$

$$\prod_{n=1}^{\infty}\left(1+\frac{1}{x_n+iy_n}\right)\left(1+\frac{1}{x_n-iy_n}\right) = \omega(-1) = 1.1662436\cdots$$

Theorem 5.2.4

Let non-trivial zeros of Dirichlet beta function are x_n+iy_n $n=1,2,3,\cdots$ and γ be Euler-Mascheroni constant. If the following expression holds, non-trivial zeros whose real parts is not $1/2$ do not exist.

$$\sum_{n=1}^{\infty}\frac{1}{1/4+y_n^2} = 4\log\Gamma\left(\frac{3}{4}\right)+\frac{\gamma}{2}+\log 2-\frac{3\log\pi}{2} = 0.07778398\cdots$$

Incidentally, when this was calculated using 10000 y_r , both sides coincided with the decimal point 3 digits.

In **5.3**, we show that $\omega(z)$ is factored completely.

Theorem 5.3.1 (Factorization of $\omega(z)$)

Let Dirichlet beta function be $\beta(z)$, the non-trivial zeros are $z_n = x_n\pm iy_n$ $n=1,2,3,\cdots$ and completed beta function be as follows.

$$\omega(z) = \left(\frac{2}{\sqrt{\pi}}\right)^{1+z}\Gamma\left(\frac{1+z}{2}\right)\beta(z)$$

Then, $\omega(z)$ is factorized as follows.

$$\omega(z) = \prod_{n=1}^{\infty}\left(1-\frac{2x_n z}{x_n^2+y_n^2}+\frac{z^2}{x_n^2+y_n^2}\right)$$

In **5.4**, we first derive the factorization of $\Omega(z)$.

Theorem 5.4.1 (Factorization of $\Omega(z)$)

Let Dirichlet beta function be $\beta(z)$, the non-trivial zeros are $z_n = x_n \pm iy_n \quad n=1,2,3,\cdots$ and completed beta function be as follows.

$$\Omega(z) = \left(\frac{2}{\sqrt{\pi}}\right)^{\frac{3}{2}+z} \Gamma\left\{\frac{1}{2}\left(\frac{3}{2}+z\right)\right\}\beta\left(\frac{1}{2}+z\right)$$

Then, $\Omega(z)$ is factorized as follows.

$$\Omega(z) = \Omega(0)\prod_{n=1}^{\infty}\left\{1-\frac{2(x_n-1/2)z}{(x_n-1/2)^2+y_n^2}+\frac{z^2}{(x_n-1/2)^2+y_n^2}\right\}$$

$$\text{Where, } \Omega(0) = \prod_{n=1}^{\infty}\frac{(x_n-1/2)^2+y_n^2}{x_n^2+y_n^2} = \left(\frac{2}{\sqrt{\pi}}\right)^{3/2}\Gamma\left(\frac{3}{4}\right)\beta\left(\frac{1}{2}\right) = 0.98071361\cdots$$

And, using this theorem and Theorem 4.2.1 in the previous section, we obtaine the following theorem.

Theorem 5.4.4

When Dirichlet beta function is $\beta(z)$ and the non-trivial zeros are $z_n = x_n \pm iy_n \quad n=1,2,3,\cdots$,
If the following expression holds, non-trivial zeros whose real parts is not $1/2$ do not exist.

$$\prod_{r=1}^{\infty}\frac{y_r^2}{1/4+y_r^2} = \left(\frac{2}{\sqrt{\pi}}\right)^{3/2}\Gamma\left(\frac{3}{4}\right)\beta\left(\frac{1}{2}\right) = 0.98071361\cdots \qquad (4.4_0)$$

Incidentally, when this was calculated using 10000 y_r , both sides coincided with the decimal point 4 digits.

06 Zeros on the Critical Line of Dirichlet Beta

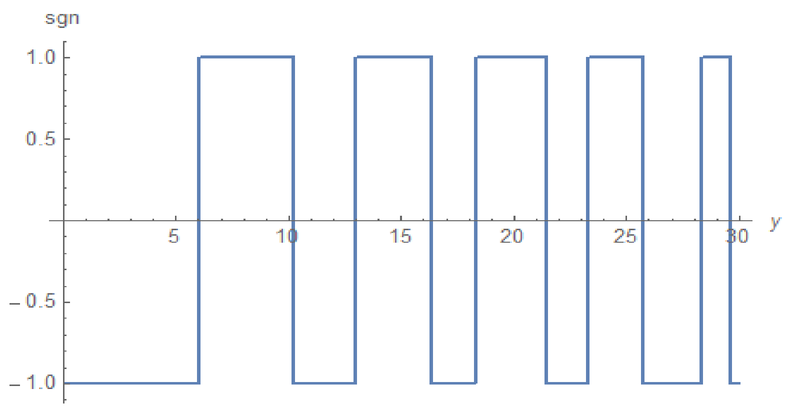
In **6.1**, substituting $z=0+iy$ for the completed Dirichlet beta $\Omega(z)$,

$$\Omega_h(z) = \left(\frac{2}{\sqrt{\pi}}\right)^{\frac{3}{2}+iy} \Gamma\left\{\frac{1}{2}\left(\frac{3}{2}+iy\right)\right\}\beta\left(\frac{1}{2}+iy\right)$$

We use this to calculate the zeros on the critical line. However, this function is too small in absolute value and can only find the zeros up to $y=917$.

So we normalize $\Omega_h(y)$ and define the following sign function.

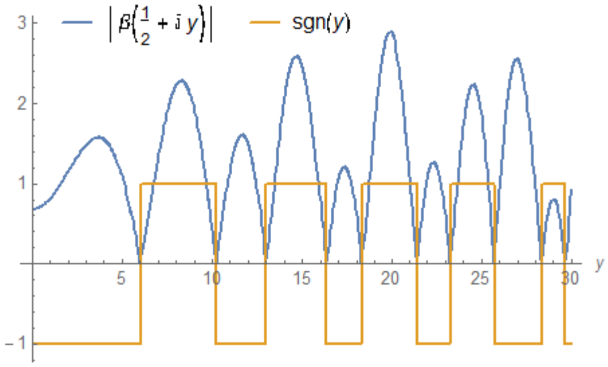
$$sgn(y) = -\frac{\Omega_h(y)}{|\Omega_h(y)|} = -\left(\frac{2}{\sqrt{\pi}}\right)^{iy}\frac{\Gamma\left\{\frac{1}{2}\left(\frac{3}{2}+iy\right)\right\}\beta\left(\frac{1}{2}+iy\right)}{\left|\Gamma\left\{\frac{1}{2}\left(\frac{3}{2}+iy\right)\right\}\beta\left(\frac{1}{2}+iy\right)\right|}$$



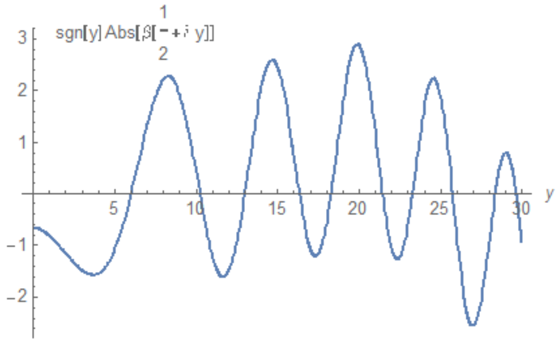
Using this sign function $sgn(y)$, we can find the zeros at large y .

In **6.2**, multiplying this sign function $sgn(y)$ by the absolute value of the Dirichlet beta $\beta(1/2+iy)$, we obtain a smooth function $B(y)$.

$$B(y) = sgn(y)\left|\beta\left(\frac{1}{2}+iy\right)\right| = -\left(\frac{2}{\sqrt{\pi}}\right)^{iy}\frac{\Gamma\left\{\frac{1}{2}\left(\frac{3}{2}+iy\right)\right\}}{\left|\Gamma\left\{\frac{1}{2}\left(\frac{3}{2}+iy\right)\right\}\right|}\beta\left(\frac{1}{2}+iy\right)$$



$\Rightarrow$



Using this $B(y)$ function, we can find the zeros on the critical line of $\beta(z)$ by the intersection of the curve and the y-axis

In **6.3**, first, a lemma is prepared.

Lemma

When $f(z)$ is a complex function defined on the domain D, the following expression holds.

$$e^{i\,Im\log f(z)} = \frac{f(z)}{|f(z)|}$$

Applying this lemma to the gamma function in the **6.2**,

$$B(y) = -\left(\frac{2}{\sqrt{\pi}}\right)^{iy}\frac{\Gamma\left\{\frac{1}{2}\left(\frac{3}{2}+iy\right)\right\}}{\left|\Gamma\left\{\frac{1}{2}\left(\frac{3}{2}+iy\right)\right\}\right|}\beta\left(\frac{1}{2}+iy\right)$$

$$= -\left(\frac{2}{\sqrt{\pi}}\right)^{iy} e^{i\,Im\log\Gamma\left\{\frac{1}{2}\left(\frac{3}{2}+iy\right)\right\}}\beta\left(\frac{1}{2}+iy\right)$$

$$= -e^{i\left[Im\log\Gamma\left\{\frac{1}{2}\left(\frac{3}{2}+iy\right)\right\} - \frac{y}{2}\log\frac{\pi}{4}\right]}\beta\left(\frac{1}{2}+iy\right)$$

From this, we obtain

$$B(y) = -e^{i\theta(y)}\beta\left(\frac{1}{2}+iy\right) \qquad \text{where,} \quad \theta(y) = Im\log\Gamma\left\{\frac{1}{2}\left(\frac{3}{2}+iy\right)\right\} - \frac{y}{2}\log\frac{\pi}{4}$$

This is Riemann-Siegel style B function .

07 Absolute Value of Dirichlet Beta Function

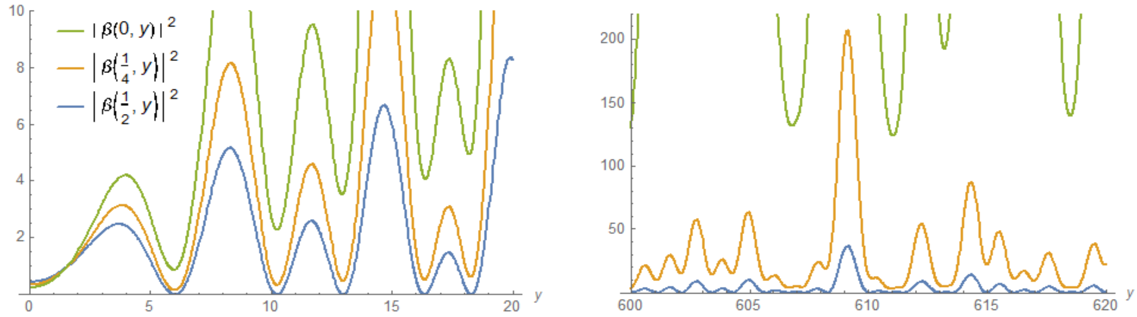
In **7.1**, Dirichlet Beta Function is observed. It is illustrated that there is no singularity and that there are non-trivial zeros in the critical strip.

In **7.2**, the properties of the square of the absolute value of Dirichlet Beta Function $|\beta(x,y)|^2$ are investigated. Then, the interval $0 \le x \le 1/2$ is particularly noted, and the following hypothesis equivalent to Riemann hypothesis is presented.

Hypothesis 7.2.1

When $\beta(x,y)$ is the Dirichlet Beta function on the complex plane, the squared absolute value $|\beta(x,y)|^2$ is a monotonically decreasing function in the region $0 < x < 1/2$, $y \ge 2$.

The figures of section in $x=0$, $1/4$, $1/2$ are drawn as follows.



In **7.3**, the square of the absolute value of Dirichlet Beta Function is expressed by a double series.

Formula 7.3.2

When $\beta(x,y)$ is the Dirichlet Beta Function,

$$|\beta(x,y)|^2 = \sum_{r=1}^{\infty}\sum_{s=1}^{\infty}\frac{(-1)^{r+s}}{\{(2r-1)(2s-1)\}^x}\cos\left(y\log\frac{2s-1}{2r-1}\right)$$

And using this, Hypothesis 7.2.1 is represented as follows.

Hypothesis 7.3.3

When $\beta(x,y)$ is the Dirichlet Beta function on the complex plane, the following inequality holds.

$$|\beta(x,y)|^2 = \sum_{r=1}^{\infty}\sum_{s=1}^{\infty}\frac{(-1)^{r+s}}{\{(2r-1)(2s-1)\}^x}\cos\left(y\log\frac{2s-1}{2r-1}\right) > 0 \quad for \quad \begin{matrix} 0 < x < 1/2 \\ y \ge 2 \end{matrix}$$

In **7.4**, the properties at the zero of the square of the absolute value of Dirichlet Beta Function are stated as theorems.

Theorem 7.4.0

When $\beta(x,y)$ is Dirichlet Beta Function, if $\beta(a,b)=0$,

$$\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}\frac{(-1)^{r+s}}{\{(2r-1)(2s-1)\}^a}\cos\left(b\log\frac{2s-1}{2r-1}\right) = 0$$

$$\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}\frac{(-1)^{r+s}}{\{(2r-1)(2s-1)\}^a}\sin\left(b\log\frac{2s-1}{2r-1}\right) = 0$$

Interestingly, at a zero point (a,b) of β , each of these rows have to be all 0 .

Theorem 7.4.1

When $\beta(x,y)$ is Dirichlet Beta Function, if $\beta(a,b)=0$,

$$\sum_{s=1}^{\infty}\frac{(-1)^{r+s}}{\{(2r-1)(2s-1)\}^a}\cos\left(b\log\frac{2s-1}{2r-1}\right) = 0 \qquad for\ \ r=1,2,3,\cdots$$

$$\sum_{s=1}^{\infty}\frac{(-1)^{r+s}}{\{(2r-1)(2s-1)\}^a}\sin\left(b\log\frac{2s-1}{2r-1}\right) = 0 \qquad for\ \ r=1,2,3,\cdots$$

From this, the following important corollary is obtained.

Corollary 7.4.1"

When $\beta(x,y)$ is Dirichlet Beta Function, if $\beta(a,b)=0$, the following expressions hold for arbitrary real number θ .

$$\sum_{s=1}^{\infty}\frac{(-1)^s}{(2s-1)^a}\cos\{b\log(2s-1)+\theta\} = 0$$

$$\sum_{s=1}^{\infty} \frac{(-1)^s}{(2s-1)^a} \sin\{b \log(2s-1) + \theta\} = 0$$

In **7.5**, the partial derivatives of the square of the absolute value are calculated.

Formula 7.5.1

When squared absolute value of Dirichlet beta function is

$$f(x,y) = \sum_{r=1}^{\infty}\sum_{s=1}^{\infty} \frac{(-1)^{r+s}}{\{(2r-1)(2s-1)\}^x} \cos\left(y \log \frac{2s-1}{2r-1}\right) \qquad (= |\beta(x,y)|^2)$$

The 1st order partial derivatives are given as follows.

$$f_x = -2 \sum_{r=1}^{\infty}\sum_{s=1}^{\infty} (-1)^{r+s} \frac{\log(2r-1)}{\{(2r-1)(2s-1)\}^x} \cos\left(y \log \frac{2s-1}{2r-1}\right)$$

$$f_y = 2 \sum_{r=1}^{\infty}\sum_{s=1}^{\infty} (-1)^{r+s} \frac{\log(2r-1)}{\{(2r-1)(2s-1)\}^x} \sin\left(y \log \frac{2s-1}{2r-1}\right)$$

Then, the necessary conditions for the square of the absolute value to be 0 are shown.

Theorem 7.5.2 (Stationary Condition)

When $\beta(x,y)$ is Dirichlet Beta Function, if $\beta(a,b) = 0$, the following expressions hold .

$$f_x(a,b) = -2 \sum_{r=1}^{\infty}\sum_{s=1}^{\infty} (-1)^{r+s} \frac{\log(2r-1)}{\{(2r-1)(2s-1)\}^a} \cos\left(b \log \frac{2s-1}{2r-1}\right) = 0$$

$$f_y(a,b) = 2 \sum_{r=1}^{\infty}\sum_{s=1}^{\infty} (-1)^{r+s} \frac{\log(2r-1)}{\{(2r-1)(2s-1)\}^a} \sin\left(b \log \frac{2s-1}{2r-1}\right) = 0$$

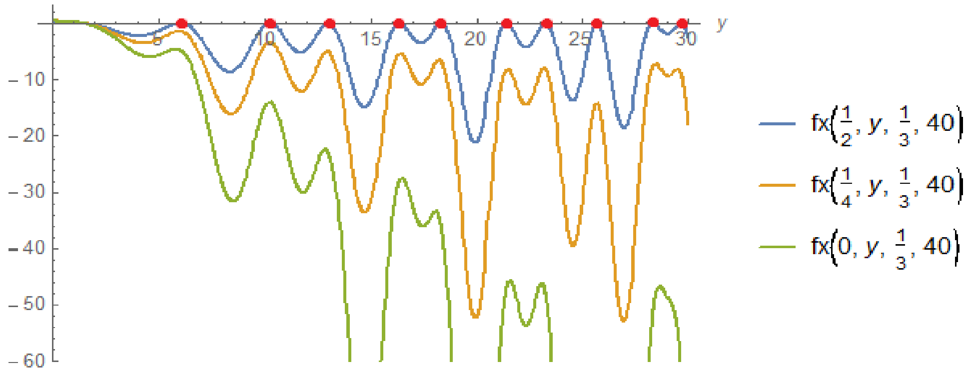
Last, using Formula 7.5.1 , a hypothesis equivalent to Riemann hypothesis is presented.

Hypothesis 7.5.5

When $\beta(x,y)$ is the Dirichlet Beta function on the complex plane, the following inequality holds.

$$f_x = -2 \sum_{r=1}^{\infty}\sum_{s=1}^{\infty} (-1)^{r+s} \frac{\log(2r-1)}{\{(2r-1)(2s-1)\}^x} \cos\left(y \log \frac{2s-1}{2r-1}\right) < 0 \quad for \quad \begin{array}{l} 0 < x < 1/2 \\ y \ge 2 \end{array}$$

this is illustrated as follows. Blue is $f_x(1/2, y)$, orange is $f_x(1/4, y)$, green is $f_x(0, y)$ and the red points are zeros on the critical line $(x = 1/2)$. Other than the blue line are not in contact with the y-axis.



Graphical Proof of the Riemann Hypothesis for the Dirichlet Beta

This proof is difficult to contain in one chapter, so I write it as a separate paper.

In **Chapter 1**, the definitions of the Dirichlet beta function $\beta(z)$ and what are known so far are explained.

Dirichle Beta Function

Dirichle Beta Function $\beta(z)$ is defined by the following Dirichlet series.

$$\beta(z) = \sum_{r=1}^{\infty} e^{-z\log(2r-1)} = \frac{1}{1^z} - \frac{1}{3^z} + \frac{1}{5^z} - \frac{1}{7^z} + - \cdots \qquad Re(z) > 1 \qquad (1.\beta)$$

This function is analytically continued to $Re(z) < 1$, and has trivial zeros $z = -(2n-1)$ (n = 1, 2, 3 , ⋯) and non-trivial zeros $z = 1/2 \pm b_n$ $(n=1,2,3,\cdots)$. So, it is the Riemann hypothesis for the Dirichlet Beta Function that there will be no non-trivial zeros other than these. In addition, it is known that non-trivial zeros exist only in the critical strip $0 < Re(z) < 1$. Also, the center line $Re(z) = 1/2$ is called the critical line .

In **Chapter 2**, three equivalent lemmas are presented and proven.

Lemma 2.1

When the set of real numbers is R and Dirichlet Beta functions is $\beta(z)$ $(z = x+iy\ ,\ x,y \in R)$, $\beta(z) = 0$ in $0 < x < 1$ if and only if the following system of equations has a solution on the domain.

$$\begin{cases} \beta(z) = \displaystyle\sum_{r=1}^{\infty} (-1)^{r-1} e^{-z\log(2r-1)} = 0 & (2.1_+) \\ \beta(1-z) = \displaystyle\sum_{r=1}^{\infty} (-1)^{r-1} e^{-(1-z)\log(2r-1)} = 0 & (2.1_-) \end{cases}$$

Lemma 2.1'

When the set of real numbers is R and Dirichlet Beta function is $\beta(z)$ $(z = x+iy\ ,\ x,y \in R)$, $\beta(1/2 \pm z) = 0$ in $-1/2 < x < 1/2$ if and only if the following system of equations has a solution on the domain.

$$\begin{cases} \beta\left(\dfrac{1}{2}+z\right) = \displaystyle\sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{\sqrt{2r-1}} e^{-z\log(2r-1)} = 0 & (2.1'_+) \\ \beta\left(\dfrac{1}{2}-z\right) = \displaystyle\sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{\sqrt{2r-1}} e^{z\log(2r-1)} = 0 & (2.1'_-) \end{cases}$$

Lemma 2.2

When the set of real numbers is R and Dirichlet Beta function is $\beta(z)$ $(z = x+iy\ ,\ x,y \in R)$, $\beta(1/2 \pm z) = 0$ in $-1/2 < x < 1/2$ if and only if the following system of equations has a solution on the domain.

$$\begin{cases} \beta_c(z) = \displaystyle\sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{\sqrt{2r-1}} \cosh\{z\log(2r-1)\} = 0 & (2.2c) \\ \beta_s(z) = \displaystyle\sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{\sqrt{2r-1}} \sinh\{z\log(2r-1)\} = 0 & (2.2s) \end{cases}$$

Then, expressing Lemma 2.2 in terms of real and imaginary parts, we obtain the following theorem.

Theorem 2.3

When the set of real numbers is R and Dirichlet Beta function is $\beta(z)$ $(z = x+iy\ ,\ x,y \in R)$, $\beta(1/2 \pm z) = 0$ in $-1/2 < x < 1/2$ if and only if the following system of equations has a solution on the domain.

$$\begin{cases} u_c(x,y) = \displaystyle\sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{\sqrt{2r-1}} \cosh\{x\log(2r-1)\}\cos\{y\log(2r-1)\} = 0 \\ v_c(x,y) = \displaystyle\sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{\sqrt{2r-1}} \sinh\{x\log(2r-1)\}\sin\{y\log(2r-1)\} = 0 \\ u_s(x,y) = \displaystyle\sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{\sqrt{2r-1}} \sinh\{x\log(2r-1)\}\cos\{y\log(2r-1)\} = 0 \\ v_s(x,y) = \displaystyle\sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{\sqrt{2r-1}} \cosh\{x\log(2r-1)\}\sin\{y\log(2r-1)\} = 0 \end{cases}$$

In **Chapter 3**, the amplitude of $v_c(x,y)$ with respect to y is studied and the following law is obtained

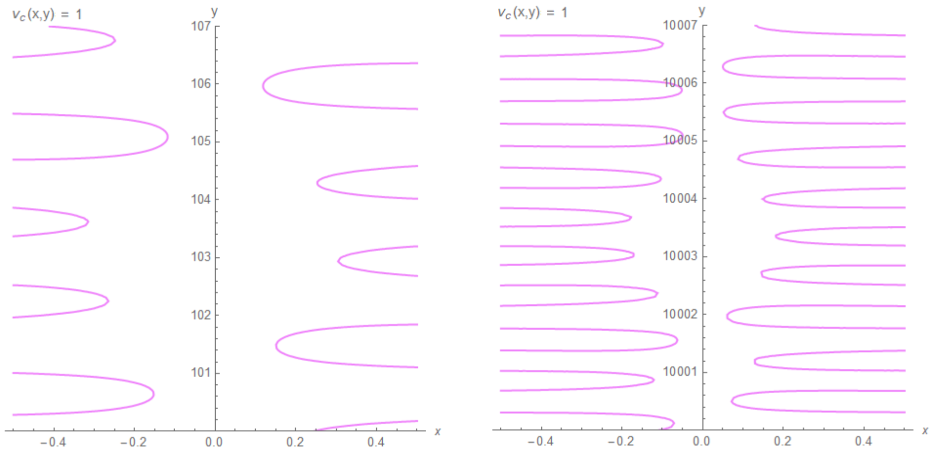
Law 3.4.5

Let x, y are real numbers and function $v_c(x,y)$ be as follows.

$$v_c(x,y) = \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{\sqrt{2r-1}} sinh\{x \log(2r-1)\} sin\{y \log(2r-1)\} \tag{2.4c}$$

Then, given x, the amplitude of $v_c(x,y)$ is generally proportional to the absolute value of y.

This law can explain that when x is given, the tips of the contour line of $v_c(x,y)$ generally approach the y-axis as y increases. For example, if a contour line of height 1 of $v_c(x,y)$ is drawn for $y=100 \sim 107$ and $y=10000 \sim 10007$, it is as follows respectively. The left figure is $y=100 \sim 107$ and the right figure is $y=10000 \sim 10007$.



In **Chapter 4**, the amplitude of $u_s(x,y)$ with respect to y is studied and the following law is obtained

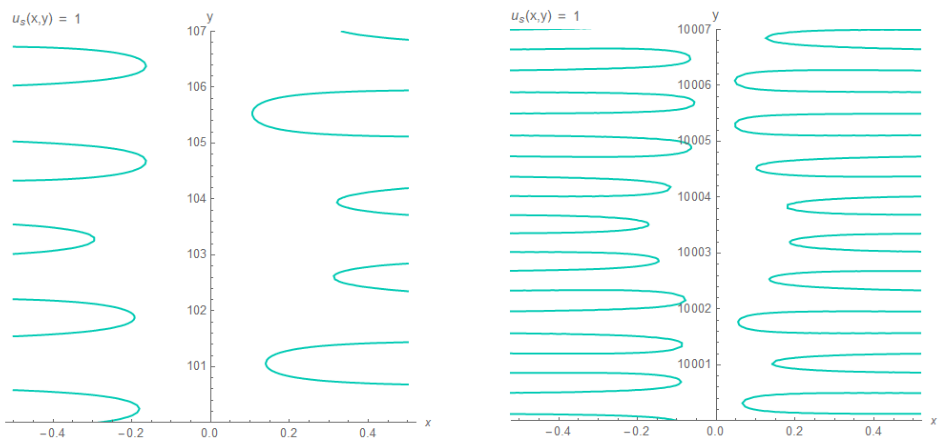
Law 4.4.5

Let x, y are real numbers and function $u_s(x,y)$ be as follows.

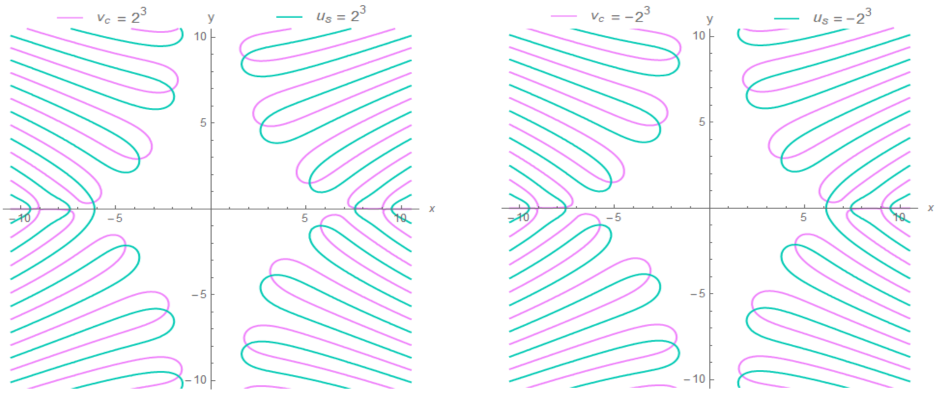
$$u_s(x,y) = \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{\sqrt{2r-1}} sinh\{x \log(2r-1)\} cos\{y \log(2r-1)\} \tag{2.4s}$$

Then, given x, the amplitude of $u_s(x,y)$ is generally proportional to the absolute value of y.

This law can explain that when x is given, the tips of the contour line of $u_s(x,y)$ generally approach the y-axis as y increases. For example, if a contour line of height 1 of $v_c(x,y)$ is drawn for $y=100 \sim 107$ and $y=10000 \sim 10007$ it is as follows respectively. The left figure is $y=100 \sim 107$ and the right figure is $y=10000 \sim 10007$.



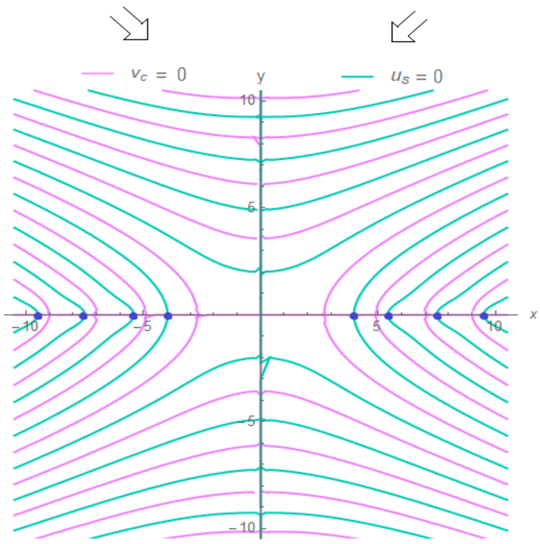
In **Chapter 5**, the contour lines of $v_c(x,y)=u_s(x,y)=\pm h\ \ (h\geq 0)$ are noticed. For example, contour lines of height ± 8 of $v_c(x,y),u_s(x,y)$ are drawn as follows. The left figure is $+8$ and the right figure is -8.



Since $v_c(x,y),u_s(x,y)$ are odd functions with respect to x, the left and right figures are mirror images with respect to the y-axis. Furthermore, since $v_c(x,y)$ is odd function with respect to y, the left and right figures are mirror images with respect to the x-axis. Both figures can never overlap by translation or rotation in the plane.

Nevertheless, at height ± 0, the left and right figures have to overlap without translation or rotation. To do this, the contour lines in both figures must deform as the height approaches ± 0 from above and below. And, at height ± 0, both figures must be symmetrical about both the y-axis and the x-axis.

This forces contour lines that were alternate at height $\neq 0$ to be opposite at height ± 0. This also applies to the x-axis. Thus, at height ± 0, the right and left edges of $\supset\ \subset$ must be absorbed into the y-axis, and the lower and upper edges of $\cup\ \cap$ must be absorbed into the x-axis. In fact, if we approach the heights of $v_c(x,y),u_s(x,y)$ from above and below to ± 0, the contour lines become eventually as follows.



For the animation of the above, click here. AnimB5218.gif

Consistent with the theory, the contour parts asymmetric with respect to the y and x-axis were absorbed in both axes. As the result, only countless trivial solutions (blue points) remained. All non-trivial solutions were absorbed on the y-axis (critical line).

Though these figures are drawn with $|y|\leq 10.5$, according to Law 3.4.5 and Law 4.4.5, the right and left tips of $\supset\ \subset$ are absorbed more quickly into the y-axis where y is large.

Thus, the system of equations $v_c(x,y)=u_s(x,y)=0$ has no solution in the critical strip $-1/2<x<1/2$ except on the critical line $x=0$.

In **Chapter 6**, by organizing and summarizing the above, the Riemann hypothesis for the Dirichlet Beta Function is proven.

Proposition 6.1 (Riemann Hypothesis)

Let $\beta(z)$ be the function defined by the following Dirichlet series.

$$\beta(z) = \sum_{r=1}^{\infty} e^{-z\log r} = \frac{1}{1^z} - \frac{1}{3^z} + \frac{1}{5^z} - \frac{1}{7^z} + - \cdots \qquad Re(z) > 1 \qquad (1.\beta)$$

This function has no non-trivial zeros except on the critical line $Re(z) = 1/2$.

Proof

According to Theorem 2.3 , the fact that the Dirichlet Beta function $\beta(z)$ has zeros in the critical strip is equivalent to the fact that the system of equations $u_c = v_c = u_s = v_s = 0$ has solutions in the critical strip. However, as seen in Chapter 5, $v_c = u_s = 0$ has no solution in the critical strip except on the critical line. Naturally, $u_c = v_c = u_s = v_s = 0$ also has no solution in the critical strip except on the critical line. Therfore, according to Theorem 2.3, the Dirichlet Beta Function $\beta(z)$ has no zeros in the critical strip except on the critical line. Q.E.D.

Analytical Proof of the Riemann Hypothesis for the Dirichlet Beta

In **Chapter 1**, the definitions of the Dirichlet beta function $\beta(z)$ etc. are stated.

Dirichlet Beta Function

Dirichle Beta Function $\beta(z)$ is defined by the following Dirichlet series.

$$\beta(z) = \sum_{r=1}^{\infty} e^{-z\log(2r-1)} = \frac{1}{1^z} - \frac{1}{3^z} + \frac{1}{5^z} - \frac{1}{7^z} + - \cdots \qquad Re(z) > 1 \qquad (1.\beta\ )$$

In **Chapter 2**, After going through three equivalent lemmas, we finally obtain the following theorem.

Theorem 2.3

When the set of real numbers is R and Dirichlet Beta Function is $\beta(z)$ $(z = x+iy\ ,\ x,y \in R)$,
$\beta(1/2 \pm z) = 0$ in $-1/2 < x < 1/2$ if and only if the following system of equations has a solution on the domain.

$$\begin{cases} u_c(x,y) = \displaystyle\sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{\sqrt{2r-1}} \cosh\{x\log(2r-1)\}\cos\{y\log(2r-1)\} = 0 \\ v_c(x,y) = \displaystyle\sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{\sqrt{2r-1}} \sinh\{x\log(2r-1)\}\sin\{y\log(2r-1)\} = 0 \\ u_s(x,y) = \displaystyle\sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{\sqrt{2r-1}} \sinh\{x\log(2r-1)\}\cos\{y\log(2r-1)\} = 0 \\ v_s(x,y) = \displaystyle\sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{\sqrt{2r-1}} \cosh\{x\log(2r-1)\}\sin\{y\log(2r-1)\} = 0 \end{cases}$$

In **Chapter 3**, it is noted that the first terms $(r = 1)$ of the $v_c(x,y)$ and $u_s(x,y)$ series are both 0. So, these first terms are changed from $r = 1$ to $r = 2$, and the following lemma is proven.

Lemma 3.1

When y is a real number, x is a real number s.t. $-1/2 < x < 1/2$, the following system of equations has no solution such that $x \neq 0$.

$$\begin{cases} v_c(x,y) = \displaystyle\sum_{r=2}^{\infty} \frac{(-1)^{r-1}}{\sqrt{2r-1}} \sinh\{x\log(2r-1)\}\sin\{y\log(2r-1)\} = 0 & (3.1c\ ) \\ u_s(x,y) = \displaystyle\sum_{r=2}^{\infty} \frac{(-1)^{r-1}}{\sqrt{2r-1}} \sinh\{x\log(2r-1)\}\cos\{y\log(2r-1)\} = 0 & (3.1s\ ) \end{cases}$$

Proof (overview)

1. Integrating the series (3.1s) term by term from 0 to y with respect to y ,

$$\int u_s(x,y)\,dy = \sum_{r=2}^{\infty} \frac{(-1)^{r-1}}{\sqrt{2r-1}\,log(2r-1)} sinh\{x\,log(2r-1)\}sin\{y\,log(2r-1)\} \qquad (3.1sy)$$

The y coordinates of the peaks and valleys of $v_c(x,y)$ and $\int u_s(x,y)\,dy$ almost match. ($\because$ $sin\{y\,log(2r-1)\}$ is shared)

The peaks and valleys of $\int u_s(x,y)\,dy$ and the zeros of $u_s(x,y)$ exactly match. ($\because$ Function and its derivative)

So, the y coordinates of the peaks and valleys of $v_c(x,y)$ and the zeros of $u_s(x,y)$ almost match.

2. Integrating the series (3.1c) term by term from 0 to y with respect to y ,

$$\int v_c(x,y)\,dy = -\sum_{r=2}^{\infty} \frac{(-1)^{r-1}}{\sqrt{2r-1}\,log(2r-1)} sinh\{x\,log(2r-1)\}cos\{y\,log(2r-1)\} \qquad (3.1cy)$$

The y coordinates of the peaks and valleys of $u_s(x,y)$ and $\int v_c(x,y)\,dy$ almost match. ($\because$ $cos\{y\,log(2r-1)\}$ is shared)

The peaks and valleys of $\int v_c(x,y)\,dy$ and the zeros of $v_c(x,y)$ exactly match. ($\because$ Function and its derivative)

So, the y coordinates of the peaks and valleys of $u_s(x,y)$ and the zeros of $v_c(x,y)$ almost match.

3. As the result of 1 and 2, $v_c(x,y)$ and $u_s(x,y)$ do not have common zeros in $-1/2 < x < 1/2$, $x \neq 0$.

In **Chapter 4**, by summarizing the above, the Riemann hypothesis for the Dirichlet Beta Function is proven.

Theorem 4.1 (Riemann Hypothesis)

Let $\beta(z)$ be the function defined by the following Dirichlet series.

$$\beta(z) = \sum_{r=1}^{\infty} e^{-z\,log\,r} = \frac{1}{1^z} - \frac{1}{3^z} + \frac{1}{5^z} - \frac{1}{7^z} + - \cdots \qquad Re(z) > 1 \qquad (1.\beta)$$

This function has no non-trivial zeros except on the critical line $Re(z) = 1/2$.

Proof (overview)

According to Lemma 3.1 and Theorem 2.3 , $\beta(1/2+z)$ has no zeros other than $x=0$ in $-1/2 < x < 1/2$.

That is, Dirichlet Beta function $\beta(z)$ has no zeros other than $x=1/2$ in $0 < x < 1$.

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